

# LOCAL WELL-POSEDNESS FOR THE $H^2$ -CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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**ABSTRACT.** In this paper, we consider the nonlinear Schrödinger equation  $iu_t + \Delta u = \lambda|u|^{\frac{4}{N-4}}u$  in  $\mathbb{R}^N$ ,  $N \geq 5$ , with  $\lambda \in \mathbb{C}$ . We prove local well-posedness (local existence, unconditional uniqueness, continuous dependence) in the critical space  $\dot{H}^2(\mathbb{R}^N)$ .

## 1. INTRODUCTION

Throughout this paper, we assume  $N \geq 5$  and consider the  $H^2$ -critical nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda|u|^\alpha u, \\ u(0) = \varphi, \end{cases} \quad (\text{NLS})$$

in  $\mathbb{R}^N$ , where  $\lambda \in \mathbb{C}$  and

$$\alpha = \frac{4}{N-4}. \quad (1.1)$$

It is often convenient to study the equivalent form equation (NLS)

$$u(t) = e^{it\Delta}\varphi - i\lambda \int_0^t e^{i(t-s)\Delta}|u(s)|^\alpha u(s) ds, \quad (1.2)$$

where  $(e^{it\Delta})_{t \in \mathbb{R}}$  is the Schrödinger group. (See, e.g., Lemma 1.1 in [10].)

Local existence for the Cauchy problem (NLS) is well known in the Sobolev space  $H^s(\mathbb{R}^N)$  provided  $\alpha < \frac{4}{N-2s}$  and (if  $s > 1$ ) that the nonlinearity is sufficiently smooth. See Kato [10], Tsutsumi [21], Cazenave and Weissler [5], Kato [12]. The smoothness condition on the nonlinearity can be improved (removed, if  $s \leq 2$ ) by estimating time derivatives of the solution instead of space derivatives. See Kato [11], Pecher [16], Fang and Han [7]. The solution depends continuously on the initial value  $H^s \rightarrow C([0, T], H^s)$ , see Kato [10], Tsutsumi [21], Cazenave, Fang and Han [4], Dai, Yang and Cao [6], Fang and Han [7]. Unconditional uniqueness (i.e., uniqueness in  $C([0, T], H^s)$  or  $L^\infty((0, T), H^s)$ , without assuming the solution belongs to some auxiliary space) is known in a number of cases, see Kato [12], Furioli and Terraneo [9], Rogers [17], Fang and Han [8]. Many of these results hold in the critical case  $\alpha = \frac{4}{N-2s}$ , see Cazenave and Weissler [5], Kato [12], Cazenave [3], Kenig and Merle [14], Tao and Visan [19], Killip and Visan [15], Cazenave, Fang and Han [4], Win and Tsutsumi [22], Fang and Han [8].

Our main result concerns the  $H^2$ -critical case (1.1), and is the following. (The homogeneous Sobolev space  $\dot{H}^2(\mathbb{R}^N)$  as well as the admissible pairs are defined in Section 2 below.)

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**Theorem 1.1.** *Suppose  $N \geq 5$ ,  $\lambda \in \mathbb{C}$  and  $\alpha$  is given by (1.1). Given any  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ , there exist a maximal existence time  $T_{\max} = T_{\max}(\varphi) > 0$  and a unique solution  $u \in C([0, T_{\max}), \dot{H}^2(\mathbb{R}^N))$  of (NLS). If, in addition,  $\varphi \in L^2(\mathbb{R}^N)$  then  $u \in C([0, T_{\max}), H^2(\mathbb{R}^N))$ . Moreover, the following properties hold.*

- (i)  $\Delta u \in L^q((0, T), L^r(\mathbb{R}^N))$  and  $u_t \in L^q((0, T), L^r(\mathbb{R}^N)) \cap C([0, T], L^2(\mathbb{R}^N))$  for every  $T < T_{\max}$  and every admissible pair  $(q, r)$ .
- (ii)  $u - e^{i\Delta} \varphi \in L^q((0, T), L^r(\mathbb{R}^N)) \cap C([0, T], L^2(\mathbb{R}^N))$  for every  $T < T_{\max}$  and every admissible pair  $(q, r)$ .
- (iii) If  $\|\Delta \varphi\|_{L^2}$  is sufficiently small, then  $T_{\max} = \infty$  and both  $u_t$  and  $\Delta u$  belong to  $L^q((0, \infty), L^r(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . If, in addition,  $\varphi \in L^2(\mathbb{R}^N)$ , then also  $u \in L^q((0, \infty), L^r(\mathbb{R}^N))$ .
- (iv) (Blowup alternative.) If  $T_{\max} < \infty$ , then  $\|u\|_{L^\gamma((0, T_{\max}), L^\nu)} = \infty$ , where  $\gamma = \frac{2(N-2)}{N-4}$ ,  $\nu = \frac{2N(N-2)}{(N-4)^2}$ .
- (v) (Continuous dependence.) Let  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ ,  $(\varphi^n)_{n \geq 1} \subset \dot{H}^2(\mathbb{R}^N)$ , let  $u$  and  $(u^n)_{n \geq 1}$  be the corresponding solutions of (NLS) and let  $T_{\max}$  and  $(T_{\max}^n)_{n \geq 1}$  denote their respective maximal existence times. Suppose  $\varphi^n \rightarrow \varphi$  in  $\dot{H}^2(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . If  $0 < T < T_{\max}$ , then  $T_{\max}^n > T$  for all sufficiently large  $n$ . Moreover,  $\Delta u^n \rightarrow \Delta u$  and  $u_t^n \rightarrow u_t$  in  $L^q((0, T), L^r(\mathbb{R}^N))$  as  $n \rightarrow \infty$ , for every admissible pair  $(q, r)$ . If, in addition,  $\varphi \in L^2(\mathbb{R}^N)$ ,  $(\varphi^n)_{n \geq 1} \subset L^2(\mathbb{R}^N)$  and  $\varphi^n \rightarrow \varphi$  in  $L^2(\mathbb{R}^N)$ , then  $u^n \rightarrow u$  in  $L^q((0, T), L^r(\mathbb{R}^N))$ .

Note that if  $u \in C([0, T], \dot{H}^2(\mathbb{R}^N))$ , then  $|u|^\alpha u \in C([0, T], L^2(\mathbb{R}^N))$  by Sobolev's embedding (see (2.7) below), so that equation (NLS) makes sense in  $L^2(\mathbb{R}^N)$ .

We note that Theorem 1.1 is the  $H^2$  counterpart of what is known in the  $H^1$ -critical case  $\alpha = \frac{4}{N-2}$ ,  $N \geq 3$ . The existence part of Theorem 1.1 in the nonhomogeneous space  $H^2(\mathbb{R}^N)$  is well-known for  $N \leq 7$  [5, Theorem 1.2], and for  $N \geq 8$  and small initial values  $\varphi$  [5, Theorem 1.4]. Local existence for large data in  $H^2$  when  $N \geq 8$ , local existence and unconditional uniqueness in the homogeneous space  $\dot{H}^2$ , and continuous dependence are new, as far as we are aware. Our proof of the existence part of Theorem 1.1 follows essentially the proof in [5], which is a fixed-point argument. The only noticeable modifications are Lemmas 2.8 and 2.9 below, which provide estimates of the nonlinear term  $|u|^\alpha u$ . These estimates replace, in our proof, the estimates given by Lemma 5.6 in [5], and allow us to remove the small data requirement in [5, Theorem 1.4]. Moreover, the set in which the fixed-point is constructed is modified, with respect to [5], in order to consider initial values in the homogeneous space  $\dot{H}^2$ . (See Definition 2.1 below.) This modification is also the key to prove property (ii) of Theorem 1.1, which means that the nonlinear term in (1.2) has better regularity properties than the solution  $u$  itself. Unconditional uniqueness follows from the argument used in [3, Proposition 4.2.5] for the  $H^1$ -critical case. Continuous dependence is established by adapting the method of [11, Theorem III']. Since we are in the critical case, a truncation argument is used, as in the proof of unconditional uniqueness.

The rest of this paper is organized as follows. In Section 2, we introduce the notation and establish a few useful estimates. In Section 3, we prove unconditional uniqueness. Section 4 is devoted to local existence and Section 5 to local continuous dependence. We complete the proof of Theorem 1.1 in Section 6. Finally, an appendix is devoted to the proof of a technical lemma (Lemma 2.4 below).

## 2. NOTATION AND PRELIMINARY RESULTS

Throughout this paper, all the function spaces we consider are made up of complex-valued functions. Given  $1 \leq p \leq \infty$ , we denote by  $p'$  the conjugate

exponent defined by  $\frac{1}{p'} = 1 - \frac{1}{p}$ . We say that a pair  $(q, r)$  is admissible if  $(q, r) \in \mathcal{A}$ , where

$$\mathcal{A} = \left\{ (q, r) \in [2, \infty] \times [2, 2N/(N-2)]; \frac{2}{q} = N \left( \frac{1}{2} - \frac{1}{r} \right) \right\}, \quad (2.1)$$

and we recall Strichartz's estimates

$$\sup_{(q,r) \in \mathcal{A}} \|e^{i \cdot \Delta} w\|_{L^q(\mathbb{R}, L^r)} \leq K \|w\|_{L^2}, \quad (2.2)$$

$$\sup_{(q,r) \in \mathcal{A}} \left\| \int_0^\cdot e^{i(\cdot-s)\Delta} f(s) ds \right\|_{L^q((0,T), L^r)} \leq K \inf_{(q',r') \in \mathcal{A}} \|f\|_{L^{q'}((0,T), L^{r'})}, \quad (2.3)$$

valid for  $0 < T \leq \infty$ , where the constant  $K$  depends only on  $N$ . Moreover, if  $w \in L^2(\mathbb{R}^N)$ , then  $e^{i \cdot \Delta} w \in C(\mathbb{R}, L^2(\mathbb{R}^N))$ ; and if the right-hand side of (2.3) is finite, then the integral on the left-hand side belongs to  $C([0, T], L^2(\mathbb{R}^N))$  if  $T < \infty$  and to  $C([0, \infty), L^2(\mathbb{R}^N))$  if  $T = \infty$ . (See [18, 13].)

It is convenient to introduce the numbers

$$\rho = \frac{2N(N-2)}{N^2 - 4N + 8}, \quad \gamma = \frac{2(N-2)}{N-4}, \quad (2.4)$$

$$\beta = \frac{2N^2}{N^2 - 2N + 8}, \quad \mu = \frac{2N}{N-4}, \quad (2.5)$$

and

$$\nu = \frac{2N(N-2)}{(N-4)^2}, \quad \theta = \frac{2N^2}{(N-2)(N-4)}. \quad (2.6)$$

It is straightforward to verify that  $(\gamma, \rho)$  and  $(\mu, \beta)$  are admissible pairs and that  $2 < \beta < \rho$ . We also recall the following Sobolev's inequalities (see [1, 20]).

$$\|u\|_{L^\mu} \leq A \|\Delta u\|_{L^2}, \quad \|u\|_{L^\nu} \leq A \|\Delta u\|_{L^\rho}, \quad \|u\|_{L^\theta} \leq A \|\Delta u\|_{L^\beta}. \quad (2.7)$$

We consider the space  $\dot{H}^2(\mathbb{R}^N)$  defined as the completion of  $\mathcal{S}(\mathbb{R}^N)$  for the norm  $\|u\|_{\dot{H}^2} = \|\Delta u\|_{L^2}$ . Alternatively, in view of (2.7),  $\dot{H}^2(\mathbb{R}^N)$  is the set of  $u \in L^\mu(\mathbb{R}^N)$  such that  $\Delta u \in L^2(\mathbb{R}^N)$ . Similarly,  $\dot{H}^{2,\rho}(\mathbb{R}^N)$  is the completion of  $\mathcal{S}(\mathbb{R}^N)$  for the norm  $\|u\|_{\dot{H}^{2,\rho}} = \|\Delta u\|_{L^\rho}$  or, equivalently, the set of  $u \in L^\nu(\mathbb{R}^N)$  such that  $\Delta u \in L^\rho(\mathbb{R}^N)$ . Note in particular that by (2.7),  $\dot{H}^2(\mathbb{R}^N) \hookrightarrow L^\mu(\mathbb{R}^N)$  and  $\dot{H}^{2,\rho}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ . As is well known, the Schrödinger group  $(e^{it\Delta})_{t \in \mathbb{R}}$  is a group of isometries on  $L^2(\mathbb{R}^N)$  and on  $\dot{H}^2(\mathbb{R}^N)$ .

We now introduce the set  $\mathcal{Y}_{\varphi,T,M}$ , in which we construct local solutions of (NLS) by a fixed-point argument (see Section 4).

**Definition 2.1.** Let  $T, M > 0$  and  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ . We denote by  $\mathcal{Y}_{\varphi,T,M}$  the set of  $u$  such that

- (i)  $u \in L^\gamma((0, T), \dot{H}^{2,\rho}(\mathbb{R}^N))$  and  $\|\Delta u\|_{L^\gamma((0,T), L^\rho)} \leq M$ ,
- (ii)  $u_t \in L^\gamma((0, T), L^\rho(\mathbb{R}^N))$  and  $\|u_t\|_{L^\gamma((0,T), L^\rho)} \leq M$ ,
- (iii)  $u - e^{i \cdot \Delta} \varphi \in L^\gamma((0, T), L^\rho(\mathbb{R}^N))$ ,
- (iv)  $u(0) = \varphi$ ,

where  $\rho$  and  $\gamma$  are given by (2.4). Moreover, we set

$$d(u, v) = \|u - v\|_{L^\gamma((0,T), L^\rho)},$$

for all  $u, v \in \mathcal{Y}_{\varphi,T,M}$ .

**Remark 2.2.** (i) Note that by Strichartz's estimate (2.2),  $\partial_t e^{i \cdot \Delta} \varphi = i e^{i \cdot \Delta} \Delta \varphi \in L^\gamma(\mathbb{R}, L^\rho(\mathbb{R}^N))$ . Thus if  $u \in \mathcal{Y}_{\varphi,T,M}$ , then  $u - e^{i \cdot \Delta} \varphi \in W^{1,\gamma}((0, T), L^\rho(\mathbb{R}^N))$ , so that  $u - e^{i \cdot \Delta} \varphi \in C([0, T], L^\rho(\mathbb{R}^N))$ . Since  $e^{i \cdot \Delta} \varphi \in C(\mathbb{R}, \dot{H}^2(\mathbb{R}^N)) \hookrightarrow C(\mathbb{R}, L^\mu(\mathbb{R}^N))$ , we see that  $u \in C([0, T], L^\rho(\mathbb{R}^N) + L^\mu(\mathbb{R}^N))$ . In particular, the condition  $u(0) = \varphi$  in Definition 2.1 makes sense in  $L^\rho(\mathbb{R}^N) + L^\mu(\mathbb{R}^N)$ .

- (ii) It is clear that  $d$  is a distance on  $\mathcal{Y}_{\varphi,T,M}$  and it is not difficult to show that  $(\mathcal{Y}_{\varphi,T,M}, d)$  is a complete metric space.

In the rest of this section, we establish useful estimates of functions in  $\mathcal{Y}_{\varphi,T,M}$ . To prove these estimates, we will use the following elementary inequalities.

**Lemma 2.3.** *Given any  $a > 0$ , there exists a constant  $C(a)$  such that*

$$| |u|^a u - |v|^a v | \leq (a+1)(|u| + |v|)^a |u - v|, \quad (2.8)$$

$$| |u|^a u - |v|^a v | \leq C(a) | |u|^{a+1} u - |v|^{a+1} v |^{\frac{a+1}{a+2}}, \quad (2.9)$$

and

$$\begin{aligned} & | |u|^a - |v|^a | + | |u|^{a-2} u^2 - |v|^{a-2} v^2 | \\ & \leq \begin{cases} C(a)(|u|^{a-1} + |v|^{a-1})|u - v| & \text{if } a \geq 1, \\ C(a)|u - v|^a & \text{if } 0 < a \leq 1, \end{cases} \end{aligned} \quad (2.10)$$

for all  $u, v \in \mathbb{C}$ .

*Proof.* Estimate (2.8) is immediate and (2.10) follows from [4, (2.26) and (2.27)]. We prove (2.9) for completeness. Let  $z \in \mathbb{C}$ ,  $|z| \leq 1$ . It follows that  $|z| \leq |z|^{\frac{1}{a+2}}$  so that  $|1 - |z|^{\frac{1}{a+2}}| = 1 - |z|^{\frac{1}{a+2}} \leq 1 - |z| = |1 - |z|| \leq |1 - z|$ ; and so,

$$|1 - |z|^{-\frac{1}{a+2}} z| \leq |1 - z| + |z|^{\frac{a+1}{a+2}} | |z|^{\frac{1}{a+2}} - 1 | \leq 2|1 - z|. \quad (2.11)$$

Since  $|1 - z| \leq 2$ , we have  $|1 - z| \leq 2^{\frac{1}{a+2}} |1 - z|^{\frac{a+1}{a+2}}$  and we deduce from (2.11) that

$$|1 - |z|^{-\frac{1}{a+2}} z| \leq 2^{\frac{a+3}{a+2}} |1 - z|^{\frac{a+1}{a+2}} \quad \text{if } |z| \leq 1. \quad (2.12)$$

Let now  $u, v \in \mathbb{C}$  with  $|v| \leq |u|$  and  $|u| \neq 0$ . Inequality (2.9) (with  $C(a) = 2^{\frac{a+3}{a+2}}$ ) follows by setting  $z = |v/u|^{a+1} (v/u)$  in (2.12) and multiplying by  $|u|^{a+1}$ .  $\square$

**Lemma 2.4.** *Let  $T > 0$ ,  $a > 0$ . Let  $q_1, q_2, r_1, r_2 \geq 1$  satisfy  $q_1, r_1 \geq a + 1$ ,  $\frac{a}{q_1} + \frac{1}{q_2} \leq 1$ ,  $\frac{a}{r_1} + \frac{1}{r_2} \leq 1$ . If  $u \in L^{q_1}((0, T), L^{r_1}(\mathbb{R}^N))$  and  $u_t \in L^{q_2}((0, T), L^{r_2}(\mathbb{R}^N))$ , then*

$$\partial_t(|u|^a u) = \frac{a+2}{2} |u|^a u_t + \frac{a}{2} |u|^{a-2} u^2 \bar{u}_t, \quad (2.13)$$

a.e. on  $(0, T) \times \mathbb{R}^N$ .

The proof of Lemma 2.4, which uses an appropriate regularization argument, is postponed to the Appendix.

**Lemma 2.5.** *Given any  $T > 0$ ,*

$$\|u\|_{L^\mu((0,T), L^\beta)} \leq \|u\|_{L^\infty((0,T), L^2)}^{\frac{2}{N}} \|u\|_{L^\gamma((0,T), L^\rho)}^{\frac{N-2}{N}}, \quad (2.14)$$

$$\|u\|_{L^{(\alpha+1)\gamma}((0,T), L^{(\alpha+1)\rho})}^{\alpha+1} \leq \|u\|_{L^\infty((0,T), L^\mu)}^\alpha \|u\|_{L^\gamma((0,T), L^\nu)}, \quad (2.15)$$

and

$$\begin{aligned} & \| |u|^\alpha u - |v|^\alpha v \|_{L^{\gamma'}((0,T), L^{\rho'})} \\ & \leq (\alpha+1)(\|u\|_{L^\gamma((0,T), L^\nu)} + \|v\|_{L^\gamma((0,T), L^\nu)})^\alpha \|u - v\|_{L^\gamma((0,T), L^\rho)}, \end{aligned} \quad (2.16)$$

hold for all functions  $u, v$  for which the right-hand side makes sense. Moreover,

$$\|\partial_t[|u|^\alpha u]\|_{L^{\gamma'}((0,T), L^{\rho'})} \leq (\alpha+1)\|u\|_{L^\gamma((0,T), L^\nu)}^\alpha \|u_t\|_{L^\gamma((0,T), L^\rho)}, \quad (2.17)$$

for all  $u \in L^\gamma((0, T), L^\nu(\mathbb{R}^N))$  such that  $u_t \in L^\gamma((0, T), L^\rho(\mathbb{R}^N))$ .

*Proof.* Both (2.14) and (2.15) follow from Hölder's inequality in space and time, by using the relations  $\frac{1}{\beta} = \frac{1}{N} + \frac{N-2}{N\rho}$  for the first one, and  $\frac{1}{\rho} = \frac{\alpha}{\mu} + \frac{1}{\nu}$  for the second one. Estimates (2.16) and (2.17) follow from Hölder's inequality in space and time and the relations  $\frac{1}{\rho'} = \frac{\alpha}{\nu} + \frac{1}{\rho}$  and  $\frac{1}{\gamma'} = \frac{\alpha}{\gamma} + \frac{1}{\gamma}$ , the first one by using (2.8) and the second one by using the inequality  $|\partial_t[|u|^\alpha u]| \leq (\alpha + 1)|u|^\alpha |u_t|$  (see (2.13)).  $\square$

**Lemma 2.6.** *Given any  $T > 0$ ,*

$$\| |u|^\alpha u \|_{L^2((0,T), L^{\frac{2N}{N-2}})} \leq A^{\alpha+1} \|\Delta u\|_{L^\infty((0,T), L^2)}^{\frac{2}{N-4}} \|\Delta u\|_{L^\gamma((0,T), L^\rho)}^{\frac{N-2}{N-4}}, \quad (2.18)$$

and

$$\begin{aligned} & \| |u|^\alpha u - |v|^\alpha v \|_{L^2((0,T), L^{\frac{2N}{N-2}})} \\ & \leq (\alpha + 1) A^{\alpha+1} \Gamma \|\Delta(u - v)\|_{L^\infty((0,T), L^2)}^{\frac{2}{N}} \|\Delta(u - v)\|_{L^\gamma((0,T), L^\rho)}^{\frac{N-2}{N}}, \end{aligned} \quad (2.19)$$

where

$$\Gamma = \|\Delta u\|_{L^\infty((0,T), L^2)}^\alpha + \|\Delta v\|_{L^\infty((0,T), L^2)}^\alpha + \|\Delta u\|_{L^\gamma((0,T), L^\rho)}^\alpha + \|\Delta v\|_{L^\gamma((0,T), L^\rho)}^\alpha,$$

hold for all  $u, v \in L^\infty((0, T), \dot{H}^2(\mathbb{R}^N)) \cap L^\gamma((0, T), \dot{H}^{2,\rho}(\mathbb{R}^N))$ .

*Proof.* Since

$$\| |u|^\alpha u \|_{L^2((0,T), L^{\frac{2N}{N-2}})} = \|u\|_{L^\mu((0,T), L^\theta)}^{\alpha+1} \leq A^{\alpha+1} \|\Delta u\|_{L^\mu((0,T), L^\beta)}^{\alpha+1},$$

by (2.7), estimate (2.18) follows from (2.14) (applied with  $u$  replaced by  $\Delta u$ ). Similarly, we deduce from (2.8) and (2.7) that

$$\begin{aligned} & \| |u|^\alpha u - |v|^\alpha v \|_{L^2((0,T), L^{\frac{2N}{N-2}})} \\ & \leq (\alpha + 1) A^{\alpha+1} [\|\Delta u\|_{L^\mu((0,T), L^\beta)}^\alpha + \|\Delta v\|_{L^\mu((0,T), L^\beta)}^\alpha] \|\Delta(u - v)\|_{L^\mu((0,T), L^\beta)}, \end{aligned}$$

and (2.19) follows by applying (2.14).  $\square$

**Lemma 2.7.** *Given any  $u, v \in \dot{H}^2(\mathbb{R}^N)$ ,*

$$\| |u|^\alpha u \|_{L^2} \leq A^{\alpha+1} \|\Delta u\|_{L^2}^{\alpha+1}, \quad (2.20)$$

and

$$\| |u|^\alpha u - |v|^\alpha v \|_{L^2} \leq (\alpha + 1) A^{\alpha+1} (\|\Delta u\|_{L^2}^\alpha + \|\Delta v\|_{L^2}^\alpha) \|\Delta(u - v)\|_{L^2}. \quad (2.21)$$

In particular, the map  $u \mapsto |u|^\alpha u$  is continuous  $\dot{H}^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ .

*Proof.* Since  $\| |u|^\alpha u \|_{L^2} = \|u\|_{L^\mu}^{\alpha+1}$ , (2.20) follows from (2.7). Similarly, we deduce from (2.8) that

$$\| |u|^\alpha u - |v|^\alpha v \|_{L^2} \leq (\alpha + 1) (\|u\|_{L^\mu}^\alpha + \|v\|_{L^\mu}^\alpha) \|u - v\|_{L^\mu},$$

and (2.21) follows by applying (2.7).  $\square$

Next, given  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ , we set

$$F(\varphi, t) = \|e^{i\cdot\Delta} \Delta \varphi\|_{L^\gamma((0,t), L^\rho)} + \|e^{i\cdot\Delta} [|\varphi|^\alpha \varphi]\|_{L^\gamma((0,t), L^\rho)}, \quad (2.22)$$

for  $0 < t \leq \infty$ . We observe that, since  $|\varphi|^\alpha \varphi \in L^2(\mathbb{R}^N)$  by (2.20),  $F$  is well defined by (2.2) and  $F(\varphi, t) \downarrow 0$  as  $t \downarrow 0$ . Moreover, it follows from (2.2) and (2.21) that the map  $(\varphi, t) \mapsto F(\varphi, t)$  is continuous  $\dot{H}^2(\mathbb{R}^N) \times (0, \infty] \rightarrow (0, \infty)$ . Therefore, if  $E$  is a compact subset of  $\dot{H}^2(\mathbb{R}^N)$ , then

$$\sup_{\varphi \in E} F(\varphi, t) \xrightarrow[t \downarrow 0]{} 0. \quad (2.23)$$

The next two lemmas are key ingredients in our proof of local existence and continuous dependence.

**Lemma 2.8.** *Let  $T, M > 0$ ,  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ , and let  $\mathcal{Y}_{\varphi,T,M}$  be as in Definition 2.1. If  $u \in \mathcal{Y}_{\varphi,T,M}$ , then*

$$\begin{aligned} \|u\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} &\leq C_1 \|\Delta\varphi\|_{L^2}^\alpha F(\varphi, T) \\ &\quad + C_1 (\|u\|_{L^\gamma((0,T),L^\nu)}^{\alpha+1} + \|u_t\|_{L^\gamma((0,T),L^\rho)}^{\alpha+1} + F(\varphi, T)^{\alpha+1}), \end{aligned} \quad (2.24)$$

where  $F$  is defined by (2.22) and the constant  $C_1$  depends only on  $N$ .

*Proof.* We follow essentially the proof of [5, Lemma 5.6]. The main difference is that we use the auxiliary function

$$v(t) = u(t) - e^{it\Delta}\varphi. \quad (2.25)$$

Observe that by the definition of  $\mathcal{Y}_{\varphi,T,M}$  and (2.2),  $v \in L^\gamma((0,T),H^{2,\rho})$ ,  $v_t \in L^\gamma((0,T),L^\rho)$ , and  $v(0) = 0$ . The crux for estimating  $v$  is the property  $v(0) = 0$ ; and  $e^{it\Delta}\varphi$  is estimated by Strichartz's estimate. We deduce from (2.15) (applied with  $u = e^{i\cdot\Delta}\varphi$ ), (2.7) and (2.22) that

$$\begin{aligned} \|e^{i\cdot\Delta}\varphi\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} &\leq A^{\alpha+1} \|\Delta\varphi\|_{L^2}^\alpha \|e^{i\cdot\Delta}\Delta\varphi\|_{L^\gamma((0,T),L^\rho)} \\ &\leq A^{\alpha+1} \|\Delta\varphi\|_{L^2}^\alpha F(\varphi, T). \end{aligned} \quad (2.26)$$

Next, it follows from (2.15) (applied with  $u = v$ ) that

$$\|v\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} \leq \|v\|_{L^\gamma((0,T),L^\nu)} \|v\|_{L^\infty((0,T),L^\mu)}^\alpha. \quad (2.27)$$

We now estimate  $\|v\|_{L^\infty((0,T),L^\mu)}$ . Since  $v(0) = 0$  and  $|\partial_t[|v|^{\gamma-1}v]| \leq \gamma|v|^{\gamma-1}|v_t|$ , we see that

$$\begin{aligned} \|v(t)\|_{L^\mu}^\gamma &= \| |v(t)|^{\gamma-1}v(t) \|_{L^{\frac{N}{N-2}}} \\ &= \left\| \int_0^t \partial_s[|v(s)|^{\gamma-1}v(s)] ds \right\|_{L^{\frac{N}{N-2}}} \\ &\leq \gamma \int_0^t \| |v|^{\gamma-1}v_s \|_{L^{\frac{N}{N-2}}} ds. \end{aligned} \quad (2.28)$$

Since  $\frac{N-2}{N} = \frac{\gamma-1}{\nu} + \frac{1}{\rho}$ , it follows from (2.28) and Hölder's inequality in space and in time that

$$\|v(t)\|_{L^\mu}^\gamma \leq \gamma \int_0^t \|v\|_{L^\nu}^{\gamma-1} \|v_t\|_{L^\rho} dt \leq \gamma \|v\|_{L^\gamma((0,T),L^\nu)}^{\gamma-1} \|v_t\|_{L^\gamma((0,T),L^\rho)},$$

for all  $0 < t < T$ , so that

$$\|v\|_{L^\infty((0,T),L^\mu)}^\alpha \leq \gamma^{\frac{2}{N-2}} \|v\|_{L^\gamma((0,T),L^\nu)}^{\frac{2N}{(N-2)(N-4)}} \|v_t\|_{L^\gamma((0,T),L^\rho)}^{\frac{2}{N-2}}. \quad (2.29)$$

It follows from (2.27) and (2.29) that

$$\|v\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} \leq \gamma^{\frac{2}{N-2}} \|v\|_{L^\gamma((0,T),L^\nu)}^{1+\frac{2N}{(N-2)(N-4)}} \|v_t\|_{L^\gamma((0,T),L^\rho)}^{\frac{2}{N-2}}. \quad (2.30)$$

Observe that by (2.25) and (2.7)

$$\begin{aligned} \|v\|_{L^\gamma((0,T),L^\nu)} &\leq \|u\|_{L^\gamma((0,T),L^\nu)} + A \|e^{i\cdot\Delta}\Delta\varphi\|_{L^\gamma((0,T),L^\rho)} \\ &\leq \|u\|_{L^\gamma((0,T),L^\nu)} + AF(\varphi, T), \end{aligned} \quad (2.31)$$

and

$$\begin{aligned} \|v_t\|_{L^\gamma((0,T),L^\rho)} &\leq \|u_t\|_{L^\gamma((0,T),L^\rho)} + \|e^{i\cdot\Delta}\Delta\varphi\|_{L^\gamma((0,T),L^\rho)} \\ &\leq \|u_t\|_{L^\gamma((0,T),L^\rho)} + F(\varphi, T). \end{aligned} \quad (2.32)$$

Note also that by (2.25)

$$\begin{aligned} \|u\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} &\leq 2^\alpha (\|e^{i\cdot\Delta}\varphi\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} + \|v\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1}). \end{aligned} \quad (2.33)$$

Estimate (2.24) follows from (2.33), (2.26), (2.30), (2.31), (2.32) and the elementary inequality  $(x+y)^{\alpha+1} \leq 2^\alpha(x^{\alpha+1} + y^{\alpha+1})$ . (Note that the various constants  $\alpha$ ,  $A$ ,  $\gamma$  only depend on  $N$ , so that  $C_1$  also only depends on  $N$ .)  $\square$

**Lemma 2.9.** *Given  $T, M > 0$  and  $\varphi, \psi \in \dot{H}^2(\mathbb{R}^N)$ , the following properties hold.*

- (i) *If  $u \in \mathcal{Y}_{\varphi,T,M}$ , then  $|u|^\alpha u \in C([0, T], L^2(\mathbb{R}^N))$ .*
- (ii) *If  $u \in \mathcal{Y}_{\varphi,T,M}$  and  $v \in \mathcal{Y}_{\psi,T,M}$ , then*

$$\begin{aligned} \| |u|^\alpha u - |v|^\alpha v \|_{L^\infty((0,T),L^2)} &\leq C_2 \left[ (\|\psi\|_{L^\mu}^{\alpha+1} + \|\varphi\|_{L^\mu}^{\alpha+1}) \|\psi - \varphi\|_{L^\mu} \right. \\ &\quad \left. + M^{\alpha+1} \|v_t - u_t\|_{L^\gamma((0,T),L^\rho)} + M^{\alpha+1} \|v - u\|_{L^\gamma((0,T),L^\nu)} \right]^{\frac{\alpha+1}{\alpha+2}}, \end{aligned} \quad (2.34)$$

where the constant  $C_2$  depends only on  $N$ .

**Remark 2.10.** Note that  $\| |u|^\alpha u \|_{L^2} = \|u\|_{L^\mu}^{\alpha+1}$ . Therefore, estimate (2.34) with  $v = 0$ ,  $\psi = 0$  implies that

$$\|u\|_{L^\infty((0,T),L^\mu)} \leq C_2^{\frac{1}{\alpha+1}} \left[ \|\varphi\|_{L^\mu}^{\alpha+2} + M^{\alpha+1} (\|u_t\|_{L^\gamma((0,T),L^\rho)} + \|u\|_{L^\gamma((0,T),L^\nu)}) \right]^{\frac{1}{\alpha+2}},$$

so that

$$\|u\|_{L^\infty((0,T),L^\mu)} \leq C_2^{\frac{1}{\alpha+1}} \left[ A^{\alpha+2} \|\Delta\varphi\|_{L^2}^{\alpha+2} + (1+A)M^{\alpha+2} \right]^{\frac{1}{\alpha+2}}, \quad (2.35)$$

for all  $u \in \mathcal{Y}_{\varphi,T,M}$ .

*Proof of Lemma 2.9.* We first prove Property (i). Note that  $\frac{\mu}{\gamma} = \frac{2(\alpha+1)}{\alpha+2}$ , so that  $|u|^\alpha u \in C([0, T], L^2(\mathbb{R}^N))$  if and only if  $w \in C([0, T], L^{\frac{\mu}{\gamma}}(\mathbb{R}^N))$ , where  $w = |u|^{\alpha+1}u$ . Note that

$$\|w\|_{L^1((0,T),L^{\frac{\mu}{\gamma}})} = \|u\|_{L^\gamma((0,T),L^\nu)}^{\alpha+2} \leq A^{\alpha+2} M^{\alpha+2}. \quad (2.36)$$

Moreover,  $\frac{\gamma}{\mu} = \frac{\alpha+1}{\nu} + \frac{1}{\rho}$ ,  $1 = \frac{\alpha+1}{\gamma} + \frac{1}{\gamma}$  and  $|w_t| \leq (\alpha+2)|u|^{\alpha+1}|u_t|$ , so that by Hölder's inequality in space and time

$$\begin{aligned} \|w_t\|_{L^1((0,T),L^{\frac{\mu}{\gamma}})} &\leq (\alpha+2) \|u\|_{L^\gamma((0,T),L^\nu)}^{\alpha+1} \|u_t\|_{L^\gamma((0,T),L^\rho)} \\ &\leq (\alpha+2) A^{\alpha+1} M^{\alpha+2}. \end{aligned} \quad (2.37)$$

Note that estimates (2.36) and (2.37) alone do not imply  $w \in C([0, T], L^{\frac{\mu}{\gamma}}(\mathbb{R}^N))$ . (Let for example  $w(t) \equiv w_0$  with  $w_0 \in L^{\frac{\mu}{\gamma}} \setminus L^{\frac{\mu}{\gamma}}.$ ) We use the property  $u(0) = \varphi$  to complete the proof of (i). Let  $X = L^{\frac{\mu}{\gamma}}(\mathbb{R}^N) + L^{\frac{\mu}{\gamma}}(\mathbb{R}^N)$ . It follows from (2.36) and (2.37) that  $w \in W^{1,1}((0, T), X) \hookrightarrow C([0, T], X)$ . In particular, there exists a sequence  $t_n \downarrow 0$  such that  $w(t_n) \rightarrow w(0)$  a.e. on  $\mathbb{R}^N$ . On the other hand,  $u(t) \rightarrow \varphi$  in  $L^\rho(\mathbb{R}^N) + L^\mu(\mathbb{R}^N)$  as  $t \downarrow 0$ , by Remark 2.2 (i). Therefore, by possibly extracting a subsequence, we deduce that  $u(t_n) \rightarrow \varphi$  a.e. on  $\mathbb{R}^N$ . Thus  $w(t_n) \rightarrow |\varphi|^{\alpha+1}\varphi$  a.e. on  $\mathbb{R}^N$  and we conclude that  $w(0) = |\varphi|^{\alpha+1}\varphi$ . Since  $\varphi \in L^{2(\alpha+1)}(\mathbb{R}^N)$  by (2.20), we conclude that  $w(0) \in L^{\frac{\mu}{\gamma}}(\mathbb{R}^N)$ . We now write

$$w(t) = w(0) + \int_0^t w_t(s) ds.$$

Since  $w_t \in L^1((0, T), L^{\frac{\mu}{\gamma}}(\mathbb{R}^N))$  by (2.37), we see that  $w \in C([0, T], L^{\frac{\mu}{\gamma}}(\mathbb{R}^N))$ , which proves Property (i).

Let now  $u, v$  be as in (ii). It follows in particular from (i) that  $|u|^{\alpha+1}u, |v|^{\alpha+1}v \in C([0, T], L^{\frac{\mu}{\gamma}}(\mathbb{R}^N))$ . Applying (2.13) with  $a = \alpha + 1$  to both  $u$  and  $v$ , we obtain

$$\begin{aligned} \partial_t(|v|^{\alpha+1}v - |u|^{\alpha+1}u) &= \frac{\alpha+3}{2}|v|^{\alpha+1}(v_t - u_t) + \frac{\alpha+3}{2}(|v|^{\alpha+1} - |u|^{\alpha+1})u_t \\ &\quad + \frac{\alpha+1}{2}|v|^{\alpha-1}v^2(\overline{v_t} - \overline{u_t}) + \frac{\alpha+1}{2}(|v|^{\alpha-1}v^2 - |u|^{\alpha-1}u^2)\overline{u_t}. \end{aligned}$$

Using (2.10) with  $a = \alpha + 1$ , we deduce that there exists a constant  $C$  depending only on  $N$  such that

$$\begin{aligned} |\partial_t(|v|^{\alpha+1}v - |u|^{\alpha+1}u)| &\leq C(|v|^{\alpha+1} + |u|^{\alpha+1})|v_t - u_t| \\ &\quad + C(|v|^\alpha + |u|^\alpha)|u_t||v - u|. \end{aligned}$$

Applying Hölder's inequality in space and in time, it follows that

$$\begin{aligned} \|\partial_t(|v|^{\alpha+1}v - |u|^{\alpha+1}u)\|_{L^1((0,T), L^{\frac{\mu}{\gamma}})} &\leq C(\|v\|_{L^\gamma((0,T), L^\nu)}^{\alpha+1} + \|u\|_{L^\gamma((0,T), L^\nu)}^{\alpha+1})\|v_t - u_t\|_{L^\gamma((0,T), L^\rho)} \\ &\quad + C(\|v\|_{L^\gamma((0,T), L^\nu)}^\alpha + \|u\|_{L^\gamma((0,T), L^\nu)}^\alpha)\|v - u\|_{L^\gamma((0,T), L^\nu)}\|u_t\|_{L^\gamma((0,T), L^\rho)}. \end{aligned}$$

We deduce by using (2.7) that

$$\begin{aligned} \|\partial_t(|v|^{\alpha+1}v - |u|^{\alpha+1}u)\|_{L^1((0,T), L^{\frac{\mu}{\gamma}})} &\leq 2C(AM)^{\alpha+1}\|v_t - u_t\|_{L^\gamma((0,T), L^\rho)} + 2CA^\alpha M^{\alpha+1}\|v - u\|_{L^\gamma((0,T), L^\nu)}, \end{aligned}$$

so that

$$\begin{aligned} \| |v|^{\alpha+1}v - |u|^{\alpha+1}u \|_{L^\infty((0,T), L^{\frac{\mu}{\gamma}})} &\leq \| |\psi|^{\alpha+1}\psi - |\varphi|^{\alpha+1}\varphi \|_{L^{\frac{2(\alpha+1)}{\alpha+2}}} \\ &\quad + 2C(AM)^{\alpha+1}\|v_t - u_t\|_{L^\gamma((0,T), L^\rho)} + 2CA^\alpha M^{\alpha+1}\|v - u\|_{L^\gamma((0,T), L^\nu)}. \end{aligned} \quad (2.38)$$

Finally, by (2.8) and Hölder's inequality

$$\| |\psi|^{\alpha+1}\psi - |\varphi|^{\alpha+1}\varphi \|_{L^{\frac{\mu}{\gamma}}} \leq (\alpha+2)(\|\psi\|_{L^\mu}^{\alpha+1} + \|\varphi\|_{L^\mu}^{\alpha+1})\|\psi - \varphi\|_{L^\mu}. \quad (2.39)$$

Since

$$\| |u|^\alpha u - |v|^\alpha v \|_{L^\infty((0,T), L^2)} \leq 2^{\frac{\alpha+3}{\alpha+2}} \| |v|^{\alpha+1}v - |u|^{\alpha+1}u \|_{L^\infty((0,T), L^{\frac{\mu}{\gamma}})}^{\frac{\alpha+1}{\alpha+2}},$$

by (2.9), estimate (2.34) follows from (2.38) and (2.39).  $\square$

### 3. UNCONDITIONAL UNIQUENESS

In this section, we prove unconditional uniqueness in  $C([0, T], \dot{H}^2(\mathbb{R}^N))$  for equation (NLS).

**Proposition 3.1.** *Let  $T > 0$ ,  $\varphi \in \dot{H}^2(\mathbb{R}^N)$  and suppose  $u^1, u^2 \in C([0, T], \dot{H}^2(\mathbb{R}^N))$  are two solutions of (1.2). It follows that  $u^1 = u^2$ .*

*Proof.* The proof is an obvious adaptation of the proof of Proposition 4.2.5 in [3]. Note first that, by Sobolev's embedding,  $|u^j|^\alpha u^j \in C([0, T], L^2(\mathbb{R}^N))$ , for  $j = 1, 2$ . Therefore, we deduce from equation (1.2) and Strichartz's estimate (2.3) that

$$u^j - e^{i\cdot\Delta}\varphi \in L^q((0, T), L^r(\mathbb{R}^N)), \quad j = 1, 2, \quad (3.1)$$

for every admissible pair  $(q, r)$ . Set now

$$S = \sup\{\tau \in [0, T]; u^1(t) = u^2(t) \text{ for } 0 \leq t \leq \tau\},$$

so that  $0 \leq S \leq T$ . Uniqueness follows if we show that  $S = T$ . Assume by contradiction that  $S < T$ . Changing  $u^1(\cdot), u^2(\cdot)$  to  $u^1(S + \cdot), u^2(S + \cdot)$ , we are reduced to the case  $S = 0$ , so that

$$\sup_{(q,r) \in \mathcal{A}} \|u^1 - u^2\|_{L^q((0,\tau), L^r)} > 0 \quad \text{for all } 0 < \tau \leq T. \quad (3.2)$$

On the other hand, it follows from (3.1) that  $u^1 - u^2 \in L^q((0, T), L^r(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . Moreover, it follows from equation (1.2) (for both  $u^1$  and  $u^2$ ) that

$$u^1(t) - u^2(t) = \int_0^t e^{i(t-s)\Delta} [|u^1(s)|^\alpha u^1(s) - |u^2(s)|^\alpha u^2(s)] ds.$$

Applying Strichartz's estimate (2.3), we deduce that

$$\sup_{(q,r) \in \mathcal{A}} \|u^1 - u^2\|_{L^q((0,\tau), L^r)} \leq K \| |u^1|^\alpha u^1 - |u^2|^\alpha u^2 \|_{L^2((0,\tau), L^{\frac{2N}{N+2}})}, \quad (3.3)$$

for every  $0 < \tau \leq T$ . On the other hand, it follows from (2.8) that

$$| |u^1|^\alpha u^1 - |u^2|^\alpha u^2 | \leq f |u^1 - u^2|,$$

where  $f = (\alpha + 1)(|u^1|^\alpha + |u^2|^\alpha)$ . Since  $u^1, u^2 \in C([0, T], L^\mu(\mathbb{R}^N))$ , we see that

$$f \in C([0, T], L^{\frac{N}{2}}(\mathbb{R}^N)). \quad (3.4)$$

Given any  $R > 0$ , we set

$$f_R = \min\{f, R\}, \quad f^R = f - f_R.$$

It is not difficult to show (by dominated convergence, using (3.4)) that

$$\|f^R\|_{L^\infty((0,T), L^{\frac{N}{2}})} =: \varepsilon_R \xrightarrow{R \rightarrow \infty} 0.$$

Moreover,

$$\|f_R\|_{L^\infty((0,T), L^N)} \leq R^{\frac{1}{2}} \|f\|_{L^\infty((0,T), L^{\frac{N}{2}})}^{\frac{1}{2}} =: C(R) < \infty,$$

for all  $R > 0$ . Therefore, given any  $0 < \tau \leq T$ ,

$$\|f^R |u^1 - u^2|\|_{L^2((0,\tau), L^{\frac{2N}{N+2}})} \leq \varepsilon_R \|u^1 - u^2\|_{L^2((0,\tau), L^{\frac{2N}{N+2}})}, \quad (3.5)$$

and

$$\begin{aligned} \|f_R |u^1 - u^2|\|_{L^2((0,\tau), L^{\frac{2N}{N+2}})} &\leq C(R) \|u^1 - u^2\|_{L^2((0,\tau), L^2)} \\ &\leq C(R) \tau^{\frac{1}{2}} \|u^1 - u^2\|_{L^\infty((0,\tau), L^2)}. \end{aligned} \quad (3.6)$$

It follows from (3.3), (3.5) and (3.6) that

$$\sup_{(q,r) \in \mathcal{A}} \|u^1 - u^2\|_{L^q((0,\tau), L^r)} \leq K[\varepsilon_R + C(R)\tau^{\frac{1}{2}}] \sup_{(q,r) \in \mathcal{A}} \|u^1 - u^2\|_{L^q((0,\tau), L^r)}. \quad (3.7)$$

We first fix  $R$  sufficiently large so that  $K\varepsilon_R \leq \frac{1}{4}$ . Then, we choose  $0 < \tau_0 \leq T$  sufficiently small so that  $KC(R)\tau_0^{\frac{1}{2}} \leq \frac{1}{4}$ , and we deduce from (3.7) that

$$\sup_{(q,r) \in \mathcal{A}} \|u^1 - u^2\|_{L^q((0,\tau_0), L^r)} = 0.$$

This contradicts (3.2) and proves uniqueness.  $\square$

## 4. THE LOCAL CAUCHY PROBLEM

In this section, we prove local existence for the equation (1.2) by a fixed-point argument. More precisely, we have the following result.

**Proposition 4.1.** *Let  $M > 0$  be sufficiently small so that*

$$|\lambda|K(\alpha+1)2^\alpha A^\alpha M^\alpha \leq \frac{1}{2}, \quad (4.1)$$

$$|\lambda|[K(\alpha+1)A^\alpha + C_1(1+A^{\alpha+1})]M^\alpha \leq \frac{1}{2}, \quad (4.2)$$

where  $C_1$  is the constant in Lemma 2.8, and  $K$  and  $A$  are the constants in (2.2)-(2.3) and (2.7), respectively. Let  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ ,  $T > 0$  and suppose further that

$$F(\varphi, T) \leq \frac{M}{4}, \quad (4.3)$$

$$(2 + |\lambda|C_1\|\Delta\varphi\|_{L^2}^\alpha)F(\varphi, T) + |\lambda|C_1F(\varphi, T)^{\alpha+1} \leq \frac{M}{2}. \quad (4.4)$$

It follows that there exists a solution  $u \in C([0, T], \dot{H}^2(\mathbb{R}^N)) \cap L^\gamma((0, T), \dot{H}^{2,\rho}(\mathbb{R}^N))$  of (1.2). Moreover,  $u \in \mathcal{Y}_{\varphi, T, M}$  (given by Definition 2.1),  $\Delta u \in L^q((0, T), L^r(\mathbb{R}^N))$  and  $u_t \in L^q((0, T), L^r(\mathbb{R}^N)) \cap C([0, T], L^2(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . If, in addition,  $\varphi \in L^2(\mathbb{R}^N)$ , then  $u \in L^q((0, T), L^r(\mathbb{R}^N)) \cap C([0, T], L^2(\mathbb{R}^N))$ .

*Proof.* We look for a fixed point of the map  $\Phi$  defined by

$$\begin{aligned} \Phi(u)(t) &= e^{it\Delta}\varphi - i\lambda \int_0^t e^{i(t-s)\Delta}[|u|^\alpha u](s) ds \\ &= e^{it\Delta}\varphi - i\lambda \int_0^t e^{is\Delta}[|u|^\alpha u](t-s) ds \end{aligned} \quad (4.5)$$

in the set  $\mathcal{Y}_{\varphi, T, M}$  of Definition 2.1. Note that  $\Phi(u)$  satisfies

$$\begin{cases} i\Phi_t + \Delta\Phi = \lambda|u|^\alpha u, \\ \Phi(0) = \varphi, \end{cases} \quad (4.6)$$

and that

$$\partial_t \Phi(u) = ie^{it\Delta}[\Delta\varphi - |\varphi|^\alpha \varphi] - i\lambda \int_0^t e^{i(t-s)\Delta} \partial_s[|u|^\alpha u](s) ds. \quad (4.7)$$

We first claim that  $\mathcal{Y}_{\varphi, T, M}$  is nonempty. Indeed, if  $\tilde{u}(t) \equiv e^{it\Delta}\varphi$ , then

$$\|\tilde{u}_t\|_{L^\gamma((0, T), L^\rho)} = \|\Delta\tilde{u}\|_{L^\gamma((0, T), L^\rho)} \leq F(T, \varphi) \leq M$$

by (4.3). Since  $\tilde{u} - e^{i\cdot\Delta}\varphi = 0$ , we see that  $\tilde{u} \in \mathcal{Y}_{\varphi, T, M}$ . Next, it follows from (2.17) and (2.7) that

$$\begin{aligned} \|\partial_t[|u|^\alpha u]\|_{L^{\gamma'}((0, T), L^{\rho'})} &\leq (\alpha+1)A^\alpha \|\Delta u\|_{L^\gamma((0, T), L^\rho)}^\alpha \|u_t\|_{L^\gamma((0, T), L^\rho)} \\ &\leq (\alpha+1)A^\alpha M^{\alpha+1}. \end{aligned} \quad (4.8)$$

Applying (4.7), (2.22) (2.3) and (4.8), we see that if  $u \in \mathcal{Y}_{\varphi, T, M}$ , then

$$\|\partial_t \Phi(u)\|_{L^\gamma((0, T), L^\rho)} \leq 2F(\varphi, T) + |\lambda|K(\alpha+1)A^\alpha M^{\alpha+1} \leq M, \quad (4.9)$$

where we used (4.3) and (4.1) in the last inequality. In view of (4.6) we have

$$\|\Delta\Phi(u)\|_{L^\gamma((0, T), L^\rho)} \leq \|\partial_t \Phi(u)\|_{L^\gamma((0, T), L^\rho)} + |\lambda|\|u\|_{L^{(\alpha+1)\gamma}((0, T), L^{(\alpha+1)\rho})}^{\alpha+1}. \quad (4.10)$$

It follows from (4.10), the first inequality in (4.9) and (2.24) that for every  $u \in \mathcal{Y}_{\varphi,T,M}$ ,

$$\begin{aligned} \|\Delta\Phi(u)\|_{L^\gamma((0,T),L^\rho)} &\leq (2 + |\lambda|C_1\|\Delta\varphi\|_{L^2}^\alpha)F(\varphi,T) + |\lambda|C_1F(\varphi,T)^{\alpha+1} \\ &\quad + |\lambda|[K(\alpha+1)A^\alpha + C_1(1+A^{\alpha+1})]M^{\alpha+1} \leq M, \end{aligned} \quad (4.11)$$

where we used (4.4) and (4.2) in the last inequality. Next, observe that  $|u|^\alpha u \in L^\infty((0,T),L^2(\mathbb{R}^N))$  by Lemma 2.9 (i). Therefore, it follows from (4.5) and Strichartz's estimate that  $\Phi(u) - e^{i\cdot\Delta}\varphi \in L^\gamma((0,T),L^\rho(\mathbb{R}^N))$ . Thus we see that  $\Phi : \mathcal{Y}_{\varphi,T,M} \mapsto \mathcal{Y}_{\varphi,T,M}$ . We next deduce from (2.16) that, given  $u, v \in \mathcal{Y}_{\varphi,T,M}$ ,

$$\||u|^\alpha u - |v|^\alpha v\|_{L^{\gamma'}((0,T),L^{\rho'})} \leq (\alpha+1)(2AM)^\alpha d(u,v);$$

and so, applying (4.5) and (2.3),

$$\|\Phi(u) - \Phi(v)\|_{L^\gamma((0,T),L^\rho)} \leq |\lambda|K(\alpha+1)(2AM)^\alpha d(u,v) \leq \frac{1}{2}d(u,v),$$

where we used (4.5) in the last inequality. Thus we see that  $d(\Phi(u), \Phi(v)) \leq \frac{1}{2}d(u,v)$  for all  $u, v \in \mathcal{Y}_{\varphi,T,M}$ , and it follows from Banach's fixed point theorem that  $\Phi$  has a fixed point  $u \in \mathcal{Y}_{\varphi,T,M}$ . In particular,  $u$  is a solution of the integral equation (1.2).

We now prove the further regularity properties. We first claim that  $u_t, \Delta u \in C([0,T],L^2(\mathbb{R}^N) \cap L^q((0,T),L^r(\mathbb{R}^N)))$  for every admissible pair  $(q,r)$ . Indeed, note that (see (4.7))

$$u_t = ie^{it\Delta}[\Delta\varphi - |\varphi|^\alpha\varphi] - i\lambda \int_0^t e^{i(t-s)\Delta} \partial_s[|u|^\alpha u](s) ds. \quad (4.12)$$

Since  $\Delta\varphi - |\varphi|^\alpha\varphi \in L^2(\mathbb{R}^N)$  and  $\partial_t[|u|^\alpha u] \in L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}^N))$  by (4.8), it follows from Strichartz's estimates that  $u_t \in L^q((0,T),L^r(\mathbb{R}^N))$  for every admissible pair  $(q,r)$  and  $u_t \in C([0,T],L^2(\mathbb{R}^N))$ . Since  $|u|^\alpha u \in C([0,T],L^2(\mathbb{R}^N))$  by Lemma 2.9 (i), it follows from the equation (NLS) that  $\Delta u \in C([0,T],L^2(\mathbb{R}^N))$ . Next, since  $\Delta u \in L^\gamma((0,T),L^\rho(\mathbb{R}^N)) \cap L^\infty((0,T),L^2(\mathbb{R}^N))$ , it follows from (2.18) that  $|u|^\alpha u \in L^2((0,T),L^{\frac{2N}{N-2}}(\mathbb{R}^N))$ . Furthermore,  $|u|^\alpha u \in L^\infty((0,T),L^2(\mathbb{R}^N))$  by Lemma 2.9 (i); and so, by applying Hölder's inequality, we see that  $|u|^\alpha u \in L^q((0,T),L^r(\mathbb{R}^N))$  for every admissible pair  $(q,r)$ . Since  $\Delta u = -iu_t + \lambda|u|^\alpha u$ , we conclude that  $\Delta u \in L^q((0,T),L^r(\mathbb{R}^N))$ .

Finally, suppose further that  $\varphi \in L^2(\mathbb{R}^N)$ . Since  $|u|^\alpha u \in L^\infty((0,T),L^2(\mathbb{R}^N))$  by Lemma 2.9 (i), we deduce from equation (1.2) and Strichartz estimates (2.2) and (2.3) that  $u \in L^q((0,T),L^r(\mathbb{R}^N)) \cap C([0,T],L^2(\mathbb{R}^N))$  for every admissible pair  $(q,r)$ . This completes the proof.  $\square$

## 5. CONTINUOUS DEPENDENCE

In this section, we prove continuous dependence on a small time interval.

**Proposition 5.1.** *Let  $M > 0$  satisfy (4.1)-(4.2) and*

$$2(\alpha+1)KA^\alpha M^\alpha \leq \frac{1}{2}, \quad (5.1)$$

$$(\alpha+1)|\lambda|KA^\alpha M^\alpha \leq \frac{1}{2}, \quad (5.2)$$

$$2(\alpha+1)(2M)^{\frac{\alpha}{\alpha+1}}(|\lambda|C_1)^{\frac{1}{\alpha+1}} \leq \frac{1}{2}, \quad (5.3)$$

where  $K$ ,  $A$  and  $C_1$  are the constant in (2.2)-(2.3), (2.7) and (2.24), respectively. Let  $(\varphi^n)_{n \geq 0} \subset \dot{H}^2(\mathbb{R}^N)$  and suppose

$$\varphi^n \xrightarrow{n \rightarrow \infty} \varphi^0 \quad \text{in } \dot{H}^2(\mathbb{R}^N). \quad (5.4)$$

Let  $T > 0$  and suppose further that

$$F(\varphi^n, T) \leq \frac{M}{4}, \quad (5.5)$$

$$(2 + |\lambda|C_1\|\Delta\varphi^n\|_{L^2}^\alpha)F(\varphi^n, T) + |\lambda|C_1F(\varphi^n, T)^{\alpha+1} \leq \frac{M}{2}, \quad (5.6)$$

for all  $n \geq 1$ . For every  $n \geq 0$ , let  $u^n \in \mathcal{Y}_{\varphi^n, T, M}$  be the solution of (1.2) with  $\varphi$  replaced by  $\varphi^n$ , given by Proposition 4.1. (The assumptions of Proposition 4.1 are satisfied, by (4.1), (4.2), (5.5) and (5.6).) It follows that  $\Delta u^n \rightarrow \Delta u^0$  and  $u_t^n \rightarrow u_t^0$  in  $L^q((0, T), L^r(\mathbb{R}^N))$  as  $n \rightarrow \infty$ , for every admissible pair  $(q, r)$ . If, in addition,  $(\varphi^n)_{n \geq 0} \subset L^2(\mathbb{R}^N)$  and  $\varphi^n \rightarrow \varphi^0$  in  $L^2(\mathbb{R}^N)$ , then  $u^n \rightarrow u^0$  in  $L^q((0, T), L^r(\mathbb{R}^N))$ .

*Proof.* Since  $u^n \in \mathcal{Y}_{\varphi^n, T, M}$ , we have

$$\|u_t^n\|_{L^\gamma((0, T), L^\rho)} \leq M, \quad \|\Delta u^n\|_{L^\gamma((0, T), L^\rho)} \leq M, \quad (5.7)$$

for all  $n \geq 0$ . We set

$$\delta_n = \sup_{(q, r) \in \mathcal{A}} \|u_t^n - u_t^0\|_{L^q((0, T), L^r)}, \quad (5.8)$$

$$\sigma_n = \|\Delta(u^n - u^0)\|_{L^\gamma((0, T), L^\rho)}, \quad (5.9)$$

where  $\mathcal{A}$  is defined by (2.1). We also set

$$\eta_n = \|\Delta(\varphi^n - \varphi^0)\|_{L^2} + \||\varphi^n|^\alpha \varphi^n - |\varphi^0|^\alpha \varphi^0\|_{L^2} + F(\varphi^n - \varphi^0, T), \quad (5.10)$$

where  $F$  is defined by (2.22). It follows from (2.21), (5.4), and Strichartz's estimate (2.2) that

$$\eta_n \xrightarrow{n \rightarrow \infty} 0. \quad (5.11)$$

We now proceed in five steps.

STEP 1. We prove that

$$\sup_{(q, r) \in \mathcal{A}} \|w^n\|_{L^q((0, T), L^r)} \xrightarrow{n \rightarrow \infty} 0, \quad (5.12)$$

where  $\mathcal{A}$  is defined by (2.1) and

$$w^n = (u^n - e^{i \cdot \Delta} \varphi^n) - (u^0 - e^{i \cdot \Delta} \varphi^0). \quad (5.13)$$

Indeed, it follows from (1.2) (for  $u^0$  and  $u^n$ ) that

$$w^n(t) = -i\lambda \int_0^t e^{i(t-s)\Delta} [|u^n(s)|^\alpha u^n(s) - |u^0(s)|^\alpha u^0(s)] ds. \quad (5.14)$$

It follows from (2.8) that

$$\begin{aligned} & ||u^n(s)|^\alpha u^n(s) - |u^0(s)|^\alpha u^0(s)| \\ & \leq (\alpha + 1)(|u^n(s)|^\alpha + |u^0(s)|^\alpha) |u^n(s) - u^0(s)| \leq g_1^n + g_2^n, \end{aligned} \quad (5.15)$$

where

$$\begin{aligned} g_1^n &= (\alpha + 1)(|u^n(s)|^\alpha + |u^0(s)|^\alpha) |w^n(s)|, \\ g_2^n &= (\alpha + 1)(|u^n(s)|^\alpha + |u^0(s)|^\alpha) |e^{is\Delta}(\varphi^n - \varphi^0)|. \end{aligned}$$

Note first that

$$\begin{aligned} \||u^\ell(s)|^\alpha |w^n(s)|\|_{L^{\gamma'}((0, T), L^{\rho'})} &\leq \|u^\ell\|_{L^\gamma((0, T), L^\rho)}^\alpha \|w^n\|_{L^\gamma((0, T), L^\rho)} \\ &\leq A^\alpha \|\Delta u^\ell\|_{L^\gamma((0, T), L^\rho)}^\alpha \|w^n\|_{L^\gamma((0, T), L^\rho)} \\ &\leq A^\alpha M^\alpha \|w^n\|_{L^\gamma((0, T), L^\rho)}, \end{aligned}$$

for all  $\ell, n \geq 0$ , by (5.7). Therefore,

$$\|g_1^n\|_{L^{\gamma'}((0, T), L^{\rho'})} \leq 2(\alpha + 1)A^\alpha M^\alpha \|w^n\|_{L^\gamma((0, T), L^\rho)}. \quad (5.16)$$

Next, since  $\frac{1}{2} = \frac{\alpha}{\mu} + \frac{1}{\mu}$ , it follows from Hölder's estimate and (2.7) that

$$\begin{aligned} \| |u^\ell(s)|^\alpha |e^{is\Delta}(\varphi^n - \varphi^0)| \|_{L^2} &\leq \|u^\ell(s)\|_{L^\mu}^\alpha \|e^{is\Delta}(\varphi^n - \varphi^0)\|_{L^\mu} \\ &\leq A \|u^\ell(s)\|_{L^\mu}^\alpha \|\Delta(\varphi^n - \varphi^0)\|_{L^2}, \end{aligned}$$

for all  $0 \leq s \leq T$  and  $\ell, n \geq 0$ . Applying (2.35) and (5.10), we obtain

$$\| |u^\ell(s)|^\alpha |e^{is\Delta}(\varphi^n - \varphi^0)| \|_{L^2} \leq AC_2^{\frac{\alpha}{\alpha+1}} \left[ A^{\alpha+2} \|\Delta\varphi^\ell\|_{L^2}^{\alpha+2} + (1+A)M^{\alpha+2} \right]^{\frac{\alpha}{\alpha+2}} \eta_n,$$

so that

$$\|g_2^n\|_{L^1((0,T),L^2)} \leq 2(\alpha+1)TAC_2^{\frac{\alpha}{\alpha+1}} \left[ A^{\alpha+2} \|\Delta\varphi^\ell\|_{L^2}^{\alpha+2} + (1+A)M^{\alpha+2} \right]^{\frac{\alpha}{\alpha+2}} \eta_n. \quad (5.17)$$

We now set  $g = |u^n(s)|^\alpha u^n(s) - |u^0(s)|^\alpha u^0(s)$ . Since  $|g| \leq g_1^n + g_2^n$  by (5.15), there exist measurable functions  $\tilde{g}_1^n$  and  $\tilde{g}_2^n$  such that  $|\tilde{g}_1^n| \leq g_1^n$ ,  $|\tilde{g}_2^n| \leq g_2^n$  and  $g = \tilde{g}_1^n + \tilde{g}_2^n$ .<sup>1</sup> Therefore, it follows from (5.14), (5.16), (5.17) and Strichartz's estimate (2.3) that

$$\begin{aligned} \sup_{(q,r)} \|w^n\|_{L^q((0,T),L^r)} &\leq 2(\alpha+1)KA^\alpha M^\alpha \|w_n\|_{L^\gamma((0,T),L^\rho)} \\ &\quad + 2(\alpha+1)TKAC_2^{\frac{\alpha}{\alpha+1}} \left[ A^{\alpha+2} \|\Delta\varphi^n\|_{L^2}^{\alpha+2} + (1+A)M^{\alpha+2} \right]^{\frac{\alpha}{\alpha+2}} \eta_n. \end{aligned}$$

Applying (5.1), (5.4) and (5.11), we conclude that (5.12) holds.

STEP 2. We prove that

$$\delta_n \xrightarrow{n \rightarrow \infty} 0, \quad (5.18)$$

where  $\delta_n$  is defined by (5.8). Indeed, we deduce from formula (4.12) and Strichartz's estimates (2.2)-(2.3) that

$$\begin{aligned} \delta_n &\leq K \|\Delta(\varphi^n - \varphi^0)\|_{L^2} + K \| |\varphi^n|^\alpha \varphi^n - |\varphi^0|^\alpha \varphi^0 \|_{L^2} \\ &\quad + |\lambda| K \|\partial_t(|u^n|^\alpha u^n - |u^0|^\alpha u^0)\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ &\leq K\eta_n + |\lambda| K \|\partial_t(|u^n|^\alpha u^n - |u^0|^\alpha u^0)\|_{L^{\gamma'}((0,T),L^{\rho'})}. \end{aligned} \quad (5.19)$$

We now apply (2.13) with  $a = \alpha$  to both  $u$  and  $v$  and we obtain

$$\begin{aligned} |\partial_t(|v|^\alpha v - |u|^\alpha u)| &\leq (\alpha+1)|v|^\alpha |v_t - u_t| \\ &\quad + \frac{\alpha+2}{2} \left[ |v|^\alpha - |u|^\alpha + |v|^{\alpha-2} v^2 - |u|^{\alpha-2} u^2 \right] |u_t|. \end{aligned}$$

Applying (2.10) with  $a = \alpha$ , we deduce that

$$|\partial_t(|v|^\alpha v - |u|^\alpha u)| \leq (\alpha+1)|v|^\alpha |v_t - u_t| + BF(u, v)|u_t|, \quad (5.20)$$

where

$$F(u, v) = \begin{cases} (|u|^{\alpha-1} + |v|^{\alpha-1})|u - v| & \text{if } \alpha \geq 1, \\ |u - v|^\alpha & \text{if } 0 < \alpha \leq 1, \end{cases} \quad (5.21)$$

and the constant  $B \geq 1$  depends only on  $N$ . It follows from (5.20) that

$$\begin{aligned} \|\partial_t(|u^n|^\alpha u^n - |u^0|^\alpha u^0)\|_{L^{\gamma'}((0,T),L^{\rho'})} &\leq (\alpha+1) \|u^n\|_{L^\gamma((0,T),L^\rho)}^\alpha \|u_t^n - u_t^0\|_{L^\gamma((0,T),L^\rho)} \\ &\quad + B \|F(u^n, u^0)|u_t^0|\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ &\leq (\alpha+1)A^\alpha M^\alpha \delta_n + B \|F(u^n, u^0)|u_t^0|\|_{L^{\gamma'}((0,T),L^{\rho'})}, \end{aligned} \quad (5.22)$$

where we used (2.7), (5.7) and (5.8) in the last inequality. Applying now (5.19) and (5.22), we see that

$$\delta_n \leq K\eta_n + (\alpha+1)|\lambda|KA^\alpha M^\alpha \delta_n + |\lambda|KB \|F(u^n, u^0)|u_t^0|\|_{L^{\gamma'}((0,T),L^{\rho'})}.$$

<sup>1</sup>For example,  $\tilde{g}_1^n = g$  if  $|g| \leq g_1^n$  and  $\tilde{g}_1^n = g_1^n |g|^{-1} g$  otherwise.

which yields, using (5.2)

$$\delta_n \leq 2K\eta_n + 2|\lambda|KB\|F(u^n, u^0)|u_t^0|\|_{L^{\gamma'}((0,T),L^{\rho'})}. \quad (5.23)$$

Now, observe that  $|u_t^0|$  is a fixed function of  $L^\gamma((0,T),L^\rho(\mathbb{R}^N))$ . Therefore, if we set

$$f_R = \min\{|u_t^0|, R\}, \quad f^R = |u_t^0| - f_R, \quad (5.24)$$

for  $R > 0$ , then

$$\|f^R\|_{L^\gamma((0,T),L^\rho)} =: \varepsilon_R \xrightarrow{R \uparrow \infty} 0, \quad (5.25)$$

and

$$\|f_R\|_{L^\gamma((0,T),L^\rho)} \leq \|u_t\|_{L^\gamma((0,T),L^\rho)} \leq M. \quad (5.26)$$

Moreover, if  $\tilde{\rho} \geq \rho$ , then

$$\|f_R(t)\|_{L^{\tilde{\rho}}}^{\tilde{\rho}} \leq R^{\tilde{\rho}-\rho} \|u_t(t)\|_{L^\rho}^\rho, \quad (5.27)$$

for a.a.  $t$ , so that

$$\|f_R\|_{L^\gamma((0,T),L^{\tilde{\rho}})} \leq R^{\frac{\tilde{\rho}-\rho}{\tilde{\rho}}} T^{\frac{\tilde{\rho}-\rho}{\gamma\tilde{\rho}}} \|u_t\|_{L^\gamma((0,T),L^\rho)}^{\frac{\rho}{\tilde{\rho}}} \leq R^{\frac{\tilde{\rho}-\rho}{\tilde{\rho}}} T^{\frac{\tilde{\rho}-\rho}{\gamma\tilde{\rho}}} M^{\frac{\rho}{\tilde{\rho}}}. \quad (5.28)$$

We now fix  $R > 0$ . Writing  $|u_t^0| = f^R + f_R$ , we see that

$$\begin{aligned} & \|F(u^n, u^0)|u_t^0|\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ & \leq \varepsilon_R \|F(u^n, u^0)^{\frac{1}{\alpha}}\|_{L^\gamma((0,T),L^\nu)}^\alpha + \|F(u^n, u^0)f_R\|_{L^{\gamma'}((0,T),L^{\rho'})}. \end{aligned} \quad (5.29)$$

Since  $|F(u^n, u^0)| \leq (|u^n| + |u^0|)^\alpha$  by (5.21), we deduce from (2.7) and (5.7) that

$$\|F(u^n, u^0)^{\frac{1}{\alpha}}\|_{L^\gamma((0,T),L^\nu)}^\alpha \leq 2^\alpha A^\alpha M^\alpha,$$

and it follows from (5.29) that

$$\begin{aligned} & \|F(u^n, u^0)|u_t^0|\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ & \leq 2^\alpha A^\alpha M^\alpha \varepsilon_R + \|F(u^n - u^0)f_R\|_{L^{\gamma'}((0,T),L^{\rho'})}. \end{aligned} \quad (5.30)$$

We now estimate the last term in (5.30), and we first assume

$$\alpha \leq 1.$$

We have

$$|F(u^n, u^0)| \leq |u^n - u^0|^\alpha \leq |w^n|^\alpha + |e^{it\Delta}(\varphi^n - \varphi^0)|^\alpha, \quad (5.31)$$

where  $w^n$  is defined by (5.13). We first estimate, using (5.26), (2.7) and (2.2)

$$\begin{aligned} \| |e^{it\Delta}(\varphi^n - \varphi^0)|^\alpha f_R \|_{L^{\gamma'}((0,T),L^{\rho'})} & \leq \| |e^{it\Delta}(\varphi^n - \varphi^0)|^\alpha \|_{L^\gamma((0,T),L^\nu)}^\alpha \|f_R\|_{L^\gamma((0,T),L^\rho)} \\ & \leq M A^\alpha \| |e^{it\Delta}(\varphi^n - \varphi^0)|^\alpha \|_{L^\gamma((0,T),L^\rho)}^\alpha \\ & \leq M A^\alpha K^\alpha \| \Delta(\varphi^n - \varphi^0) \|_{L^2}^\alpha, \end{aligned}$$

so that

$$\| |e^{it\Delta}(\varphi^n - \varphi^0)|^\alpha f_R \|_{L^{\gamma'}((0,T),L^{\rho'})} \leq M A^\alpha K^\alpha \eta_n^\alpha. \quad (5.32)$$

To estimate the contribution of the second term in (5.31), we set

$$\tilde{\rho} = \frac{2(N-2)(N-4)}{N^2 - 8N + 8}.$$

It follows that  $\tilde{\rho} > \rho$  and that  $\frac{1}{\gamma'} = \frac{\alpha+1}{\gamma}$ ,  $\frac{1}{\rho'} = \frac{\alpha}{\rho} + \frac{1}{\tilde{\rho}}$ . Applying Hölder's inequality in space and time, and (5.28), we deduce that

$$\begin{aligned} \| |w^n|^\alpha f_R \|_{L^{\gamma'}((0,T),L^{\rho'})} & \leq \|w^n\|_{L^\gamma((0,T),L^\rho)}^\alpha \|f_R\|_{L^\gamma((0,T),L^{\tilde{\rho}})} \\ & \leq R^{\frac{\tilde{\rho}-\rho}{\tilde{\rho}}} T^{\frac{\tilde{\rho}-\rho}{\gamma\tilde{\rho}}} \|w^n\|_{L^\gamma((0,T),L^\rho)}^\alpha. \end{aligned} \quad (5.33)$$

It now follows from (5.23), (5.30), (5.31), (5.32), and (5.33) that

$$\begin{aligned} \delta_n &\leq 2^{\alpha+1}|\lambda|KBA^\alpha M^\alpha \varepsilon_R + 2K\eta_n \\ &\quad + 2|\lambda|MA^\alpha K^{\alpha+1}B\eta_n^\alpha + 2|\lambda|KBR^{\frac{\tilde{\rho}-\rho}{\rho}}T^{\frac{\tilde{\rho}-\rho}{\gamma\rho}}\|w^n\|_{L^\gamma((0,T),L^\rho)}^\alpha. \end{aligned} \quad (5.34)$$

We first let  $n \rightarrow \infty$  in (5.34). Applying (5.11) and (5.12), we obtain

$$\limsup_{n \rightarrow \infty} \delta_n \leq 2^{\alpha+1}|\lambda|KBA^\alpha M^\alpha \varepsilon_R.$$

Since  $R > 0$  is arbitrary, we may let  $R \rightarrow \infty$ , and (5.18) follows by using (5.25). We now suppose

$$\alpha > 1,$$

and we have

$$\begin{aligned} |F(u^n, u^0)| &\leq (|u^n|^{\alpha-1} + |u^0|^{\alpha-1})|u^n - u^0| \\ &\leq (|u^n|^{\alpha-1} + |u^0|^{\alpha-1})(|w^n| + |e^{it\Delta}(\varphi^n - \varphi^0)|). \end{aligned} \quad (5.35)$$

We set

$$\tilde{\rho} = \frac{2N(N-2)}{N^2 - 8N + 16}.$$

It follows that  $\tilde{\rho} > \rho$ , and that  $\frac{1}{\gamma'} = \frac{\alpha+1}{\gamma}$ ,  $\frac{1}{\rho'} = \frac{\alpha-1}{\nu} + \frac{1}{\rho} + \frac{1}{\tilde{\rho}}$ . We estimate by Hölder's inequality in space and time

$$\begin{aligned} &\|(|u^n|^{\alpha-1} + |u^0|^{\alpha-1})|e^{it\Delta}(\varphi^n - \varphi^0)|f_R\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ &\leq C(\|u^n\|_{L^\gamma((0,T),L^\nu)}^{\alpha-1} + \|u^0\|_{L^\gamma((0,T),L^\nu)}^{\alpha-1}) \\ &\quad \|e^{it\Delta}(\varphi^n - \varphi^0)\|_{L^\gamma((0,T),L^\nu)}\|f_R\|_{L^\gamma((0,T),L^\rho)}, \end{aligned}$$

and

$$\begin{aligned} &\|(|u^n|^{\alpha-1} + |u^0|^{\alpha-1})|w^n|f_R\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ &\leq C(\|u^n\|_{L^\gamma((0,T),L^\nu)}^{\alpha-1} + \|u^0\|_{L^\gamma((0,T),L^\nu)}^{\alpha-1}) \\ &\quad \|w^n\|_{L^\gamma((0,T),L^\rho)}\|f_R\|_{L^\gamma((0,T),L^{\tilde{\rho}})}, \end{aligned}$$

and it is not difficult to conclude as above that (5.18) holds.

STEP 3. We prove that

$$\sigma_n \xrightarrow{n \rightarrow \infty} 0, \quad (5.36)$$

where  $\sigma_n$  is defined by (5.9). Indeed, it follows from the equation (NLS) (for  $u$  and  $u^n$ ) that

$$\sigma_n \leq \delta_n + |\lambda| \| |u^n|^\alpha u^n - |u^0|^\alpha u^0 \|_{L^\gamma((0,T),L^\rho)}. \quad (5.37)$$

Note that by (2.8)

$$\begin{aligned} &\| |u^n|^\alpha u^n - |u^0|^\alpha u^0 \|_{L^\gamma((0,T),L^\rho)} \leq (\alpha+1)(\|u^n\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^\alpha \\ &\quad + \|u^0\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^\alpha) \|u^n - u^0\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}. \end{aligned} \quad (5.38)$$

Next, it follows from equation (NLS) and (5.7) that

$$|\lambda| \|u^n\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} \leq 2M, \quad (5.39)$$

for all  $n \geq 0$ . Moreover,  $u^n - u^0 \in \mathcal{Y}_{\varphi^n - \varphi^0, T, 2M}$ . Therefore, (2.24) yields the estimate

$$\|u^n - u^0\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})}^{\alpha+1} \leq C_1[2\eta_n^{\alpha+1} + A^{\alpha+1}\sigma_n^{\alpha+1} + \delta_n^{\alpha+1}].$$

Since  $(x^{\alpha+1} + y^{\alpha+1} + z^{\alpha+1})^{\frac{1}{\alpha+1}} \leq x + y + z$ , we deduce that

$$\|u^n - u^0\|_{L^{(\alpha+1)\gamma}((0,T),L^{(\alpha+1)\rho})} \leq C_1^{\frac{1}{\alpha+1}}[2^{\frac{1}{\alpha+1}}\eta_n + A\sigma_n + \delta_n]. \quad (5.40)$$

It follows from (5.38), (5.39) and (5.40) that

$$\begin{aligned} |\lambda| \| |u^n|^\alpha u^n - |u^0|^\alpha u^0 \|_{L^\gamma((0,T),L^\rho)} \\ \leq 2(\alpha+1)(2M)^{\frac{\alpha}{\alpha+1}} (|\lambda|C_1)^{\frac{1}{\alpha+1}} [2^{\frac{1}{\alpha+1}}\eta_n + A\sigma_n + \delta_n]. \end{aligned} \quad (5.41)$$

Estimates (5.37) and (5.41) yield

$$\sigma_n \leq \delta_n + 2(\alpha+1)(2M)^{\frac{\alpha}{\alpha+1}} (|\lambda|C_1)^{\frac{1}{\alpha+1}} [2^{\frac{1}{\alpha+1}}\eta_n + A\sigma_n + \delta_n]. \quad (5.42)$$

Applying (5.3), we deduce from (5.42) that  $\sigma_n \leq 3\delta_n + 2\eta_n$ , and (5.36) follows from (5.18) and (5.11).

STEP 4. We prove that

$$\Delta u^n \xrightarrow{n \rightarrow \infty} \Delta u^0 \quad \text{in } L^q((0,T), L^r(\mathbb{R}^N)), \quad (5.43)$$

for every admissible pair  $(q, r)$ . Indeed, note first that by lemma 2.9 (ii), together with Sobolev's inequality (2.7), (5.4), (5.18) and (5.36),

$$\| |u^n|^\alpha u^n - |u^0|^\alpha u^0 \|_{L^\infty((0,T),L^2)} \xrightarrow{n \rightarrow \infty} 0. \quad (5.44)$$

Next, observe that by the equation (NLS), (5.18) and (5.44),  $\Delta u^n$  is bounded, as  $n \rightarrow \infty$ , in  $L^\infty((0,T), L^2(\mathbb{R}^N))$ . Therefore, it follows from (2.19) and (5.36) that

$$\| |u^n|^\alpha u^n - |u^0|^\alpha u^0 \|_{L^2((0,T), L^{\frac{2N}{N-2}})} \xrightarrow{n \rightarrow \infty} 0. \quad (5.45)$$

It now follows from (5.44) and (5.45) that

$$\| |u^n|^\alpha u^n - |u^0|^\alpha u^0 \|_{L^q((0,T), L^r)} \xrightarrow{n \rightarrow \infty} 0, \quad (5.46)$$

for every admissible pair  $(q, r)$ . Property (5.43) follows from (5.18), (5.46) and the equation (NLS).

STEP 5. The case  $\varphi^n \rightarrow \varphi^0$  in  $L^2(\mathbb{R}^N)$ . If  $(\varphi^n)_{n \geq 0} \subset L^2(\mathbb{R}^N)$  and  $\varphi^n \rightarrow \varphi^0$  in  $L^2(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , then by Strichartz's estimate,

$$\| e^{i\Delta}(\varphi^n - \varphi^0) \|_{L^q((0,T), L^r)} \xrightarrow{n \rightarrow \infty} 0,$$

for every admissible pair  $(q, r)$ . Applying (5.12), we conclude that

$$\| u^n - u^0 \|_{L^q((0,T), L^r)} \xrightarrow{n \rightarrow \infty} 0,$$

for every admissible pair  $(q, r)$ . This completes the proof.  $\square$

## 6. PROOF OF THEOREM 1.1

In this section, we complete the proof of Theorem 1.1. Note first that uniqueness follows from Proposition 3.1.

Fix  $M > 0$  sufficiently small so that (4.1) and (4.2) are satisfied. Given an initial value  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ , it follows from (2.23) that if  $T > 0$  is sufficiently small, then (4.3) and (4.4) are satisfied. Therefore, it follows from Proposition 4.1 that there exists a solution  $u \in C([0, T], \dot{H}^2(\mathbb{R}^N))$  of (1.2). We now extend  $u$  to a maximal existence interval by the usual procedure. We set

$$T_{\max} = \sup\{\tau > 0; \text{ there exists a solution } C([0, \tau], \dot{H}^2(\mathbb{R}^N)) \text{ of (1.2)}\},$$

and it follows from what precedes that  $T_{\max} \geq T > 0$ . By uniqueness, there exists a solution  $u \in C([0, T_{\max}), \dot{H}^2(\mathbb{R}^N))$  of (1.2). We now fix  $0 < S < T_{\max}$  and show that  $u \in L^q((0, S), \dot{H}^{2,r}(\mathbb{R}^N))$  and  $u_t \in L^q((0, S), L^r(\mathbb{R}^N)) \cap C([0, S], L^2(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . Indeed, since  $u \in C([0, S], \dot{H}^2(\mathbb{R}^N))$ , we see that

$\cup_{0 \leq t \leq S} \{u(t)\}$  is a compact subset of  $\dot{H}^2(\mathbb{R}^N)$ . It then follows from (2.23) that if  $T > 0$  is sufficiently small, then

$$\sup_{0 \leq t \leq S} F(u(t), T) \leq \frac{M}{4}, \quad (6.1)$$

$$\sup_{0 \leq t \leq S} (2 + |\lambda|C_1 \|\Delta \varphi\|_{L^2}^\alpha) F(u(t), T) + |\lambda|C_1 F(u(t), T)^{\alpha+1} \leq \frac{M}{2}. \quad (6.2)$$

Therefore, we may apply Proposition 4.1 with  $\varphi$  replaced by  $u(t)$  for every  $t \in [0, S]$ . By uniqueness, we conclude easily that  $u$  has the desired regularity properties. Next, it follows from Lemma 2.9 (i) that  $|u|^\alpha u \in L^\infty((0, S), L^2(\mathbb{R}^N))$ , so that the further regularity property (ii) follows from Strichartz's estimate (2.3). If, in addition,  $\varphi \in L^2(\mathbb{R}^N)$ , then  $e^{i\Delta} \varphi \in C([0, T_{\max}), L^2(\mathbb{R}^N))$ . Therefore,  $u \in C([0, T_{\max}), L^2(\mathbb{R}^N))$ , and we conclude that  $u \in C([0, T_{\max}), H^2(\mathbb{R}^N))$ .

So far, we have proved the first statements of Theorem 1.1, as well as properties (i) and (ii). We now prove property (iii), and we fix  $M > 0$  sufficiently small so that (4.1) and (4.2) are satisfied. We note that by (2.2) and (2.20),

$$F(\varphi, \infty) \leq K[\|\Delta \varphi\|_{L^2} + A^{\alpha+1} \|\Delta \varphi\|_{L^2}^{\alpha+1}].$$

Therefore, if  $\|\Delta \varphi\|_{L^2}$  is sufficiently small, then

$$F(\varphi, \infty) \leq \frac{M}{4}, \quad (6.3)$$

$$(2 + |\lambda|C_1 \|\Delta \varphi\|_{L^2}^\alpha) F(\varphi, \infty) + |\lambda|C_1 F(\varphi, \infty)^{\alpha+1} \leq \frac{M}{2}. \quad (6.4)$$

We fix such a  $\varphi$  and we let  $u \in C([0, T_{\max}), \dot{H}^2(\mathbb{R}^N))$  be the corresponding solution of (NLS). Given any  $0 < T < \infty$ , it follows from (6.3)-(6.4) that we may apply Proposition 4.1. We therefore obtain a solution of (NLS)  $u^T \in C([0, T], \dot{H}^2(\mathbb{R}^N)) \cap \mathcal{Y}_{\varphi, T, M}$  with  $\partial_t u^T \in C([0, T], L^2(\mathbb{R}^N))$ . By uniqueness and maximality of  $T_{\max}$ , we see that  $T_{\max} > T$  and that  $u = u^T$  on  $[0, T]$ . Since  $u^T \in \mathcal{Y}_{\varphi, T, M}$ , we have  $\|\Delta u\|_{L^\gamma((0, T), L^\rho)} \leq M$  and  $\|u_t\|_{L^\gamma((0, T), L^\rho)} \leq M$ . Therefore, by the blowup alternative we see that  $T_{\max} = \infty$ . Thus, we may let  $T \rightarrow \infty$  and we see that  $\Delta u \in L^\gamma((0, \infty), L^\rho(\mathbb{R}^N))$  and  $u_t \in L^\gamma((0, \infty), L^\rho(\mathbb{R}^N))$ . Next, we deduce from (4.8) that  $\partial_t[|u|^\alpha u] \in L^{\gamma'}((0, \infty), L^{\rho'}(\mathbb{R}^N))$ , so that by (4.12) and Strichartz's estimates  $u_t \in L^q((0, \infty), L^r(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . Furthermore, we deduce from Lemma 2.9 (ii) that

$$|u|^\alpha u \in L^\infty((0, \infty), L^2(\mathbb{R}^N)), \quad (6.5)$$

and it follows from (NLS) that  $\Delta u \in L^\infty((0, \infty), L^2(\mathbb{R}^N))$ . Applying (2.18), we deduce that  $|u|^\alpha u \in L^2((0, \infty), L^{\frac{2N}{N-2}}(\mathbb{R}^N))$ . Interpolating with (6.5), we conclude that  $|u|^\alpha u \in L^q((0, \infty), L^r(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . Since  $\Delta u = -iu_t + \lambda|u|^\alpha u$ , we see that  $\Delta u \in L^q((0, \infty), L^r(\mathbb{R}^N))$ .

We now prove the blowup alternative (iv). Suppose by contradiction that  $T_{\max} < \infty$  and

$$\|u\|_{L^\gamma((0, T_{\max}), L^\rho)} < \infty. \quad (6.6)$$

We first show that

$$\|u_t\|_{L^\gamma((0, T_{\max}), L^\rho)} < \infty, \quad (6.7)$$

$$\|\Delta u\|_{L^\gamma((0, T_{\max}), L^\rho)} < \infty. \quad (6.8)$$

Fix  $\varepsilon > 0$  sufficiently small so that

$$(\alpha + 1)|\lambda|K\varepsilon^\alpha \leq \frac{1}{2}. \quad (6.9)$$

By (6.6), there exists  $T_\varepsilon \in [0, T_{\max})$  such that

$$\|u\|_{L^\gamma((T_\varepsilon, T_{\max}), L^\nu)} \leq \varepsilon.$$

Changing  $u(\cdot)$  to  $u(T_\varepsilon + \cdot)$  and  $\varphi$  to  $u(T_\varepsilon)$ , we may assume that  $T_\varepsilon = 0$ , so that

$$\|u\|_{L^\gamma((0, T_{\max}), L^\nu)} \leq \varepsilon. \quad (6.10)$$

We next observe that by (4.12), Strichartz's estimates (2.2)-(2.3), (2.20) and (2.17),

$$\begin{aligned} \|u_t\|_{L^\gamma((0, T), L^\rho)} &\leq K(\|\Delta\varphi\|_{L^2} + A^{\alpha+1}\|\Delta\varphi\|_{L^2}^{\alpha+1}) \\ &\quad + (\alpha+1)|\lambda|K\|u\|_{L^\gamma((0, T), L^\nu)}^\alpha\|u_t\|_{L^\gamma((0, T), L^\rho)}, \end{aligned}$$

for all  $0 < T < T_{\max}$ . Applying (6.10) and (6.9), we deduce that

$$\|u_t\|_{L^\gamma((0, T), L^\rho)} \leq 2K(\|\Delta\varphi\|_{L^2} + A^{\alpha+1}\|\Delta\varphi\|_{L^2}^{\alpha+1}) \quad (6.11)$$

for all  $0 < T < T_{\max}$ . Thus  $\|u_t\|_{L^\gamma((0, T_{\max}), L^\rho)} < \infty$  and (6.7) holds. We deduce from the equation (NLS) that if  $0 < T < T_{\max}$ , then

$$\|\Delta u\|_{L^\gamma((0, T), L^\rho)} \leq \|u_t\|_{L^\gamma((0, T), L^\rho)} + |\lambda|\|u\|_{L^{(\alpha+1)\gamma}((0, T), L^{(\alpha+1)\rho})}^{\alpha+1}, \quad (6.12)$$

for every  $0 < T < T_{\max}$ . It follows from (6.12), (6.7), (6.6) and (2.24) that (6.8) holds.

Next, (6.7), (6.8) and (2.17) imply that  $\partial_t[|u|^\alpha u] \in L^{\gamma'}((0, T_{\max}), L^{\rho'}(\mathbb{R}^N))$ , so that by (4.12) and Strichartz,  $u_t \in C([0, T_{\max}], L^2(\mathbb{R}^N))$ . Since also  $|u|^\alpha u \in C([0, T_{\max}], L^2(\mathbb{R}^N))$  by (6.7), (6.8) and Lemma 2.9 (i), we deduce from equation (NLS) that  $\Delta u \in C([0, T_{\max}], L^2(\mathbb{R}^N))$ , so that  $u \in C([0, T_{\max}], \dot{H}^2(\mathbb{R}^N))$ . Thus we may apply Proposition 4.1 and construct a solution  $v$  of (1.2) with  $\varphi$  replaced by  $u(T_{\max})$ , on some time interval  $[0, T]$  with  $T > 0$ . Setting

$$\tilde{u}(t) = \begin{cases} u(t) & 0 \leq t \leq T_{\max}, \\ v(t - T_{\max}) & T_{\max} \leq t \leq T_{\max} + T, \end{cases}$$

it is not difficult to see that  $\tilde{u}$  is a solution of (1.2) on  $[0, T_{\max} + T]$ , which contradicts the maximality of  $T_{\max}$  and proves the blowup alternative.

It remains to prove the continuous dependence property (v). This follows from Proposition 5.1 and a standard compactness argument. More precisely, let  $\varphi \in \dot{H}^2(\mathbb{R}^N)$ , and let  $u$  be the corresponding solution of (1.2), defined on the maximal interval  $[0, T_{\max}(\varphi))$ . Fix  $T < T_{\max}$ , and fix  $M > 0$  satisfying (4.1), (4.2), (5.1), (5.2) and (5.3). Since  $\cup_{0 \leq t \leq T} \{u(t)\}$  is a compact subset of  $\dot{H}^2(\mathbb{R}^N)$ , it follows from (2.23) that we may fix  $\tau > 0$  sufficiently small so that

$$\sup_{0 \leq t \leq T} F(u(t), \tau) \leq \frac{M}{8}, \quad (6.13)$$

$$\sup_{0 \leq t \leq T} (2 + |\lambda|C_1\|\Delta u(t)\|_{L^2}^\alpha)F(u(t), \tau) + |\lambda|C_1F(u(t), \tau)^{\alpha+1} \leq \frac{M}{4}. \quad (6.14)$$

Let  $\ell \geq 1$  be an integer such that  $(\ell - 1)\tau < T \leq \ell\tau$ . Suppose the sequence  $(\varphi^n)_{n \geq 1} \subset \dot{H}^2(\mathbb{R}^N)$  satisfies  $\varphi^n \rightarrow \varphi$  in  $\dot{H}^2(\mathbb{R}^N)$  as  $n \rightarrow \infty$  and let  $u^n$  be the corresponding solutions of (1.2), with maximal existence time  $T_{\max}(\varphi^n)$ . Since  $\varphi^n \rightarrow \varphi$ , it follows from (6.13)-(6.14) that there exists  $n_1$

$$F(\varphi^n, \tau) \leq \frac{M}{4},$$

$$(2 + |\lambda|C_1\|\Delta \varphi^n\|_{L^2}^\alpha)F(\varphi^n, \tau) + |\lambda|C_1F(\varphi^n, \tau)^{\alpha+1} \leq \frac{M}{2},$$

for all  $n \geq n_1$ . Therefore, we may apply Proposition 5.1, and it follows that  $T_{\max}(\varphi^n) > \tau$  for  $n \geq n_1$  and  $\Delta u^n \rightarrow \Delta u$  and  $u_t^n \rightarrow u_t$  in  $L^q((0, \tau), L^r(\mathbb{R}^N))$  for

every admissible pair  $(q, r)$ . If  $\tau < T$ , we deduce in particular that  $u^n(\tau) \rightarrow u(\tau)$  in  $\dot{H}^2(\mathbb{R}^N)$ , so that by (6.13)-(6.14) there exists  $n_2$  such that

$$F(u^n(\tau), \tau) \leq \frac{M}{4},$$

$$(2 + |\lambda|C_1\|\Delta u^n(\tau)\|_{L^2}^\alpha)F(u^n(\tau), \tau) + |\lambda|C_1F(u^n(\tau), \tau)^{\alpha+1} \leq \frac{M}{2},$$

for all  $n \geq n_2$ . Applying Proposition 5.1, we deduce that  $T_{\max}(\varphi^n) > 2\tau$  for  $n \geq n_2$  and  $\Delta u^n \rightarrow \Delta u$  and  $u_t^n \rightarrow u_t$  in  $L^q((0, 2\tau), L^r(\mathbb{R}^N))$  for every admissible pair  $(q, r)$ . We see that we can iterate this argument in order to cover the interval  $[0, T]$ . Finally, if  $(\varphi^n)_{n \geq 1} \subset L^2(\mathbb{R}^N)$  and  $\varphi^n \rightarrow \varphi$  in  $L^2(\mathbb{R}^N)$ , we obtain  $u^n \rightarrow u$  in  $L^q((0, T), L^r(\mathbb{R}^N))$  for every admissible pair  $(q, r)$  by applying, at each step, the corresponding statement in Proposition 5.1.

#### APPENDIX A. PROOF OF LEMMA 2.4

We give the proof of Lemma 2.4. It relies on the following property.

**Lemma A.1.** *Fix a function  $\rho \in C_c^\infty(\mathbb{R}^{N+1})$ ,  $\rho \geq 0$  with  $\|\rho\|_{L^1(\mathbb{R}^{N+1})} = 1$  and, given any  $n \geq 1$ , set  $\rho_n(t, x) = n^{N+1}\rho(nt, nx)$  for  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^N$ . Let  $1 \leq q, r < \infty$ ,  $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ , and set  $u_n = \rho_n \star u$  (where the convolution is on  $\mathbb{R}^{N+1}$ ). It follows that*

$$\|u_n\|_{L^q(\mathbb{R}, L^r)} \leq \|u\|_{L^q(\mathbb{R}, L^r)}, \quad (\text{A.1})$$

and that  $u_n \rightarrow u$  in  $L^q(\mathbb{R}, L^r(\mathbb{R}^N))$  as  $n \rightarrow \infty$ .

*Proof.* We denote by  $\star_x$  the convolution on  $\mathbb{R}^N$ . We first prove that, given any  $f \in L^1(\mathbb{R}^{N+1})$  and  $g \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ ,

$$\|f \star g\|_{L^q(\mathbb{R}, L^r)} \leq \|f\|_{L^1(\mathbb{R}^{N+1})} \|g\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^N))}. \quad (\text{A.2})$$

Indeed,

$$\begin{aligned} [f \star g](t, x) &= \int_{\mathbb{R}} \int_{\mathbb{R}^N} f(t-s, x-y) g(s, y) dy ds \\ &= \int_{\mathbb{R}} [f(t-s, \cdot) \star_x g(s, \cdot)](x) ds. \end{aligned}$$

Therefore, by Young's inequality for the convolution on  $\mathbb{R}^N$ ,

$$\|[f \star g](t, \cdot)\|_{L^r(\mathbb{R}^N)} \leq \int_{\mathbb{R}} \|f(t-s, \cdot)\|_{L^1(\mathbb{R}^N)} \|g(s, \cdot)\|_{L^r(\mathbb{R}^N)} ds.$$

We now apply Young's inequality for the convolution in time, and we deduce that

$$\|f \star g\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^N))} \leq \|f\|_{L^1(\mathbb{R}, L^1(\mathbb{R}^N))} \|u\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^N))}.$$

Inequality (A.2) follows, since  $\|f\|_{L^1(\mathbb{R}, L^1(\mathbb{R}^N))} = \|f\|_{L^1(\mathbb{R}^{N+1})}$ . Estimate (A.1) is an immediate consequence of (A.2), since  $\|\rho_n\|_{L^1(\mathbb{R}^{N+1})} = \|\rho\|_{L^1(\mathbb{R}^{N+1})} = 1$ . The convergence property follows from (A.1) and a standard density argument, see e.g. the proof of Theorem 4.22 in [2]. Note that this argument uses the density of  $C_c(\mathbb{R}^{N+1})$  in  $L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ . One can show this as follows. By the classical truncation argument,  $C_c(\mathbb{R}, L^r(\mathbb{R}^N))$  is dense in  $L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ . Then, given a function  $u \in C_c(\mathbb{R}, L^r(\mathbb{R}^N))$ , the set  $\cup_{t \in \mathbb{R}} \{u(t)\}$  is a compact subset of  $L^r(\mathbb{R}^N)$ . Therefore, by the standard truncation and convolution argument (in  $\mathbb{R}^N$ ),  $u$  can be approximated in  $L^\infty(\mathbb{R}, L^r(\mathbb{R}^N))$  by functions of  $C_c(\mathbb{R}^{N+1})$ .  $\square$

**Remark A.2.** Note that the proof of (A.2) shows the more general inequality

$$\|f \star g\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^N))} \leq \|f\|_{L^{q_1}(\mathbb{R}, L^{r_1}(\mathbb{R}^N))} \|g\|_{L^{q_2}(\mathbb{R}, L^{r_2}(\mathbb{R}^N))},$$

where  $1 \leq q, q_1, q_2, r, r_1, r_2 \leq \infty$  satisfy  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - 1$  and  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} - 1$ .

*Proof of Lemma 2.4.* For a smooth function  $u$ , identity (2.13) follows from straightforward calculations. For  $u$  as in the statement of Lemma 2.4, we extend  $u$  and  $u_t$  to  $\mathbb{R} \times \mathbb{R}^N$  by setting

$$\tilde{u} = \begin{cases} u & \text{on } (0, T) \times \mathbb{R}^N, \\ 0 & \text{elsewhere,} \end{cases} \quad \tilde{v} = \begin{cases} u_t & \text{on } (0, T) \times \mathbb{R}^N, \\ 0 & \text{elsewhere,} \end{cases}$$

and we consider the sequence  $(\rho_n)_{n \geq 1}$  given by Lemma A.1. We set  $\tilde{u}_n = \rho_n \star \tilde{u}$ ,  $\tilde{v}_n = \rho_n \star \tilde{v}$  and we note that  $\rho \in C_c^\infty(\mathbb{R}^{N+1})$ , so that  $\tilde{u}_n, \tilde{v}_n \in C^\infty(\mathbb{R}^{N+1})$ . We now fix  $0 < \varepsilon < \frac{1}{2}$  and we set  $K_\varepsilon = (\varepsilon, 1 - \varepsilon) \times \mathbb{R}^N$ . We note that for  $n \geq n_0$  with  $n_0$  sufficiently large, the convolutions giving  $\tilde{u}_n(x)$  and  $\tilde{v}_n(x)$  for  $x \in K_\varepsilon$  only see the values of  $u$  and  $u_t$  in  $(0, T) \times \mathbb{R}^N$ . Thus we see that  $\partial_t \tilde{u}_n = \tilde{v}_n$  in  $K_\varepsilon$  for  $n \geq n_0$ . Applying formula (2.13) to  $\tilde{u}$ , we deduce that

$$\partial_t(|\tilde{u}_n|^a \tilde{u}_n) = \frac{a+2}{2} |\tilde{u}_n|^a \tilde{v}_n + \frac{a}{2} |\tilde{u}_n|^{a-2} \tilde{u}_n^2 \overline{\tilde{v}_n} \quad (\text{A.3})$$

in  $K_\varepsilon$ . We now define  $q, r \geq 1$  by  $\frac{a}{q_1} + \frac{1}{q_2} = \frac{1}{q}$  and  $\frac{a}{r_1} + \frac{1}{r_2} = \frac{1}{r}$ . Applying Lemma A.1 to both  $\tilde{u}$  and  $\tilde{v}$ , then Hölder's inequality in space and time, we deduce that  $|\tilde{u}_n|^a \tilde{u}_n \rightarrow |u|^a u$  in  $L^{\frac{q_1}{a+1}}((\varepsilon, T - \varepsilon), L^{\frac{r_1}{a+1}}(\mathbb{R}^N))$  and  $\frac{a+2}{2} |\tilde{u}_n|^a \tilde{v}_n + \frac{a}{2} |\tilde{u}_n|^{a-2} \tilde{u}_n^2 \overline{\tilde{v}_n} \rightarrow \frac{a+2}{2} |u|^a u_t + \frac{a}{2} |u|^{a-2} u^2 \overline{u_t}$  in  $L^q((\varepsilon, T - \varepsilon), L^r(\mathbb{R}^N))$ , as  $n \rightarrow \infty$ . By possibly extracting a subsequence, we may assume that convergence also holds a.e. in  $K_\varepsilon$ . Letting  $n \rightarrow \infty$  in (A.3) we deduce that (2.13) holds a.e. in  $K_\varepsilon$ . Since  $0 < \varepsilon < \frac{1}{2}$  is arbitrary, we conclude that (2.13) holds a.e. in  $(0, T) \times \mathbb{R}^N$ .  $\square$

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